Russell Tracy Crawford

THE COLLECTED MATHEMATICAL WORKS

OF

GEORGE WILLIAM HILL

VOLUME ONE

PUBLISHED BY THE CARNEGIE INSTITUTION OF WASHINGTON
JUNE, 1905

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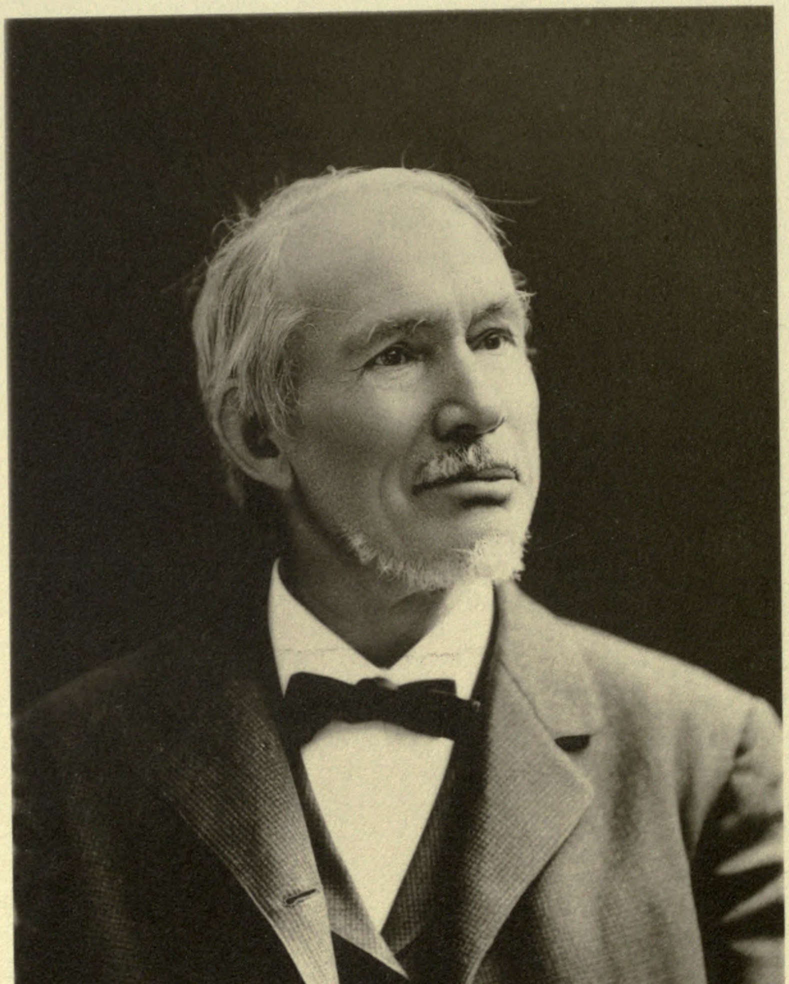

G. W. Hill

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CONTENTS.

INTRODUCTION

PAR M. H. POINCARÉ

M. Hill est une des physionomies les plus originales du monde scientifique américain. Tout entier à ses travaux et à ses calculs, il reste étranger à la vie fiévreuse qui s'agite autour de lui, il recherche l'isolement, hier dans son bureau du Nautical Almanac, aujourd'hui dans sa ferme tranquille de la vallée de l'Hudson. Cette réserve, j'allais dire cette sauvagerie, a été une circonstance heureuse pour la science, puisqu'elle lui a permis de mener jusqu'au bout ses ingénieuses et patientes recherches, sans en être distrait par les incessants accidents du monde extérieur. Mais elle a empêché que sa réputation se répandit rapidement au dehors; des années se sont écoulées avant qu'il eût, dans l'opinion du public savant, la place à laquelle il avait droit. Sa modestie ne s'en chagrinait pas trop et il ne demandait qu'une chose, le moyen de travailler en paix.

M. Hill est né à New York le 3 mars 1838. Son père, d'origine anglaise, était venu en Amérique en 1820 à l'âge de 8 ans; sa mère, d'une vieille famille huguenote, lui apportait les traditions des premiers colons de la terre américaine.

Quoique né dans une grande ville, M. Hill est un campagnard; peu de temps après sa naissance, son père quitta New York et vint s'établir à West Nyack, N. Y.; c'était une ferme, près de la rivière Hudson, à 25 milles environ de la grande Cité. C'est là que M. Hill passa son enfance; il aima toujours cette résidence; il y revenait toutes les fois qu'il le pouvait, et quand il eut quitté le Nautical Almanac, c'est encore là qu'il s'établit définitivement; c'est là qu'il poursuit tranquillement ses travaux, évitant le plus qu'il peut les voyages à New York.

Ses aptitudes exceptionnelles pour les mathématiques ne tardèrent pas à se manifester et on décida de l'envoyer au collége. En octobre 1855, à l'âge de 17 ans, il entra au Collége Rutgers, New Brunswick, N. J. Son professeur de mathématiques était le Dr. Strong, ami de M. Bowditch, le traducteur de la Mécanique Céleste de Laplace.

Le Dr. Strong était un homme de tradition, un laudator temporis acti; pour lui Euler était le Dieu des Mathématiques, et après lui la décadence avait commencé; il est vrai que c'est là un dieu que l'on peut adorer avec

profit. De rares exceptions près, la bibliothèque du Dr. Strong était impitoyablement fermée à tous les livres postérieurs à 1840. Heureusement on a écrit d'excellentes choses sur la Mécanique Céleste avant 1840; on trouvait là Laplace, Lagrange, Poisson, Pontécoulant. Tels furent les maîtres par lesquels Hill fut initié au rudiment.

En juillet 1859 il reçut ses degrés au Collége Rutgers et se rendit à Cambridge, Mass., dans l'espoir d'accroître ses connaissances mathématiques, mais il n'y resta pas longtemps, car au printemps de 1861 il obtint un poste d'assistant aux bureaux du Nautical Almanac à Washington. Il resta au service de cette éphéméride pendant trente années de sa vie, les plus fructueuses au point de vue de la production scientifique.

Les bureaux du Nautical Almanac étaient à cette époque à Cambridge (Massachusetts), où ils pouvaient profiter des ressources scientifiques de l'Université Harvard et ils étaient dirigés par M. Runkle. Ce savant avait fondé un journal de mathématiques élémentaires, The Mathematical Monthly, dans le but de favoriser les études mathématiques en Amérique en facilitant la publication de courts articles et en proposant des prix pour la solution de problèmes mathématiques. L'un des premiers articles publiés révélait la main d'un maître, et gagna aisément le prix. Il s'agissait des fonctions de Laplace et de la figure de la Terre. L'auteur était M. Hill, qui venait de sortir du collége.

C'est ainsi que l'attention de M. Runkle fut attirée sur ce jeune homme et qu'il songea à utiliser ses services pour les calculs de l'éphéméride américaine.

On l'autorisa néanmoins à continuer sa résidence dans sa maison familiale de West Nyack (village qui s'appelait alors Nyack Turnpike). Il y resta encore quand en 1886 les bureaux du Nautical Almanac furent transférés à Washington.

Mais en 1877 M. Simon Newcomb prit la direction de l'éphéméride. Il voulut entreprendre une tâche colossale, la reconstruction des tables de toutes les planètes; la part de M. Hill était la plus difficile; c'était la théorie de Jupiter et de Saturne, dont il avait commencé à s'occuper depuis 1872. Il ne pouvait la mener à bien qu'auprès de son chef et de ses collègues. Il fallut donc se résigner à l'exil; l'importance de l'œuvre à accomplir lui fit facilement accepter ce sacrifice.

Ses services furent hautement appréciés; en 1874 il fut élu membre de l'Académie Nationale des Sciences. En 1887 la Société Royale Astronomique de Londres lui accorda sa médaille d'or pour ses recherches sur la théorie de la Lune. Il fut président de la Société Mathématique Américaine pendant les années 1894 et 1895. L'université de Cambridge (Angleterre)

lui conféra des degrés honoraires, et il en fut de même de plusieurs universités américaines.

En 1892 il prit sa retraite et quitta les bureaux du Nautical Almanac; il eut hâte de s'installer pour ses dernières années dans cette chère maison où il avait passé son enfance ; au début, il la quittait encore plusieurs fois par semaine pour venir professer à l'Université Columbia à New York; mais il ne tarda pas à se lasser de cet enseignement et depuis il y vit seul avec ses livres et ses souvenirs.

Le travail quotidien du Nautical Almanac, qui est fort absorbant, lui laissait cependant assez de temps pour ses recherches originales, dont quelques-unes portent sur des objets étrangers à ses études habituelles. Dans les premières années surtout, on trouve fréquemment son nom dans ces recueils périodiques, où les amateurs de mathématiques pures se proposent de petits problèmes et se complaisent dans l'élégance des solutions, par exemple, dans "The Analyst."

Mais il ne tarda pas à se spécialiser. Non seulement ses fonctions l'y contraignaient, mais ses goûts l'y portaient. Le travail courant, nécessaire pour la préparation de l'éphéméride, lui fournissait déjà des occasions de se distinguer. Nous citerons des tables pour faciliter le calcul des positions des étoiles fixes et qui sont précédées d'une note de M. Hill où la théorie de cette réduction est exposée d'une façon simple et claire.

A cette époque le prochain passage de Vénus préoccupait tous les astronomes. En vue des expéditions projetées, le bureau de l'éphéméride dut se livrer à de longs travaux préliminaires. M. Hill fut ainsi conduit à refaire les tables de Vénus. C'était son premier ouvrage de longue haleine, et on peut y voir déjà le germe des qualités que l'on admirera plus tard dans tous ses écrits. Dans cette première période de sa vie scientifique, il revint à plusieurs reprises sur le calcul des orbites. C'est là un problème qui se présente constamment au calculateur astronomique et qui devait naturellement retenir l'attention d'un praticien constamment aux prises avec les difficultés qu'il fait naître. Citons une élégante discussion de l'équation fondamentale de Gauss et diverses notes relatives au même sujet. Les progrès de l'astronomie d'observation avaient d'ailleurs fait entrer la question dans une phase nouvelle; les découvertes de petites planètes se multiplient et deviennent de plus en plus fréquentes. Elles se succèdent avec une telle rapidité que les calculateurs sont distancés par les observateurs. Ceux-ci fournissent aux premiers plus de besogne qu'ils n'en peuvent faire, et ils veulent être servis promptement, parce que dès qu'une nouvelle planète est découverte ils craignent de la perdre. La question aujourd'hui est donc avant tout de faire vite ; il faut des méthodes rapides, qui n'exigent pas de trop longs cal-

culs et permettent d'utiliser les premières observations. On a été ainsi conduit à négliger d'abord l'excentricité des ellipses et à calculer des orbites circulaires. Tel est le point de vue où s'est placé M. Hill dans une série de notes qui ont paru dans divers recueils entre 1870 et 1874.

Mais j'ai hâte d'arriver à son œuvre capitale, à celle où s'est dévoilée toute l'originalité de son esprit, à sa théorie de la Lune. Pour en bien faire comprendre la portée, il faut d'abord rappeler quel était l'état de cette théorie au moment où M. Hill commença à s'en occuper.

Deux œuvres de haute sagacité et de longue patience venaient d'être menées à bonne fin ; je veux parler de celle de Hansen et de celle de Delaunay. Le premier, par une voie inutilement détournée, était arrivé le premier au but, devançant de beaucoup ceux qui avaient pris la bonne route. Ce phénomène, au premier abord inexplicable, n'étonnera pas beaucoup les psychologues. Si sa méthode, qui nous paraît si rébarbative, ne l'effrayait pas, c'est précisément parce qu'il était infiniment patient, et c'est pour cela aussi qu'il est allé jusqu'au bout. Et c'est aussi parce qu'elle était étrange qu'elle lui semblait avoir un cachet d'originalité, et c'est dans le sentiment de cette originalité qu'il a puisé la foi solide qui l'a soutenu dans son entreprise. Une autre raison de son succès, c'est qu'il n'a cherché que des valeurs purement numériques des coëfficients sans se préoccuper d'en trouver l'expression analytique ; ce qui chez les autres représentaient de longues formules, se réduisait pour lui à un chiffre, et cela dès le début du calcul.

Quoi qu'il en soit, c'est encore sur les tables de Hansen que nous vivons et il est probable que les nouvelles théories plus savantes, plus satisfaisantes pour l'esprit, ne donneront pas des chiffres très différents.

Delaunay est à l'extrême opposé ; ses inégalités se présentent sous la forme de formules algébriques; dans ces formules ne figurent que des lettres et des coëfficients numériques formés par le quotient de deux nombres entiers exactement calculés. Il n'a donc pas fait seulement la théorie de la Lune, mais la théorie de tout satellite qui tournerait ou pourrait tourner autour de n'importe quelle planète. A ce point de vue il laisse Hansen loin derrière lui. La méthode qui l'avait conduit à ce résultat constituait le progrès le plus important qu'eût fait la Mécanique Céleste depuis Laplace. Perfectionnée aujourd'hui et allégée, elle est devenue un instrument que chacun peut manier et qui a rendu déjà bien des services dans toutes les parties de l'Astronomie. Telle que Delaunay l'avait d'abord conçue, elle était d'un emploi plus pénible. Peut-être aurait-il abrégé considérablement son travail s'il en avait fait un usage moins exclusif, mais il faut beaucoup pardonner aux inventeurs.

Il mena à bonne fin sa tâche d'algébriste, mais les formules demandaient à être réduites en chiffres; quand un accident imprévu l'enleva à ses admirateurs, il était sur le point de commencer ces nouveaux calculs. Sa mort arrêta ce travail, et ce n'est que dans ces derniers temps qu'il put être repris et terminé.

Malheureusement les séries de Delaunay ne convergent qu'avec une désespérante lenteur. Elles procèdent suivant les puissances des excentricités de l'inclinaison, de la parallaxe du soleil, et de la quantité que l'on appelle m et qui est le rapport des moyens mouvements. Cette quantité est de $\frac{1}{13}$ environ, et si les coëfficients numériques allaient en décroissant, la convergence serait suffisante. Malheureusement il n'en est pas ainsi, ces coëfficients croissent, au contraire, très rapidement par suite de la présence de petits diviseurs. Aussi désespérant de pousser assez loin le calcul des séries, Delaunay fut-il obligé d'ajouter *au jugé* des termes complémentaires.

M. Hill s'assimila promptement la méthode de Delaunay, et en a fait l'objet de plusieurs de ses écrits, mais celle qu'il proposa était tout à fait différente et très originale. C'est dans un mémoire de l'American Journal of Mathematics, tome 1, que nous en voyons les premiers germes.

Les séries de Delaunay, nous l'avons dit, dépendent de cinq constantes, qui sont les excentricités, l'inclinaison, la parallaxe du soleil et enfin la quantité m. Si nous supposons que les quatre premières sont nulles, nous aurons une solution particulière de nos équations différentielles. Cette solution particulière sera beaucoup plus simple que la solution générale, puisque la plupart des inégalités auront disparu, et qu'une seule d'entre elles subsistera, celle qui est connue sous le nom de variation. D'autre part cette solution particulière ne représente pas exactement la trajectoire de la Lune, mais elle peut servir de première approximation, puisque les excentricités, l'inclinaison et la parallaxe sont effectivement très petites. Le choix de cette première approximation est beaucoup plus avantageux que celui de l'ellipse Keplerienne, puisque pour cette ellipse le périgée est fixe, tandis que pour l'orbite réelle il est mobile.

Les équations différentielles sont d'ailleurs elles-mêmes plus simples, puisque l'excentricité et la parallaxe étant nulles, le Soleil est supposé décrire une circonférence de rayon très grand. M. Hill simplifie encore ces équations par un choix judicieux des variables. Il prend non pas les coordonnées polaires, mais les coordonnées rectangulaires, et c'est là un grand progrès. Que ces dernières soient plus simples à tout égard, c'est de toute évidence, et cependant les astronomes répugnent à les adopter. Je comprends à la rigueur cette répugnance pour la Lune, puisque ce que nous observons, ce que nous avons besoin de calculer c'est la longitude, mais j'avoue que je me

l'explique difficilement en ce qui concerne les planètes, puisque ce n'est pas la longitude héliocentrique, mais la longitude géocentrique qu'on observe. En tous cas, pour la Lune, elle-même, M. Hill a jugé que les avantages l'emportent sur les inconvénients, et qu'on peut bien se résigner à faire à la fin du calcul un petit changement de coordonnées, pour ne pas traîner pendant toute une théorie, un encombrant bagage de variables incommodes.

Les variables de M. Hill ne sont pas d'ailleurs des coordonnées rectangulaires par rapport à des axes fixes, mais par rapport à des axes mobiles animés d'une rotation uniforme, égale à la vitesse angulaire moyenne du Soleil. D'où une simplification nouvelle, car le temps ne figure plus explicitement dans les équations. Mais l'avantage le plus important est le suivant.

Pour un observateur lié à ces axes mobiles, la Lune paraîtrait décrire une courbe fermée, si les excentricités, l'inclinaison et la parallaxe étaient nulles. Comme les équations différentielles sont d'ailleurs rigoureuses, *c'était là le premier exemple d'une solution périodique du problème des* 3 *corps*, dont l'existence était rigoureusement démontrée. Depuis ces solutions périodiques ont pris une importance tout à fait capitale en Mécanique Céleste. Mais l'auteur ne se borna pas à démontrer cette existence, il étudia dans le détail cette orbite (ou plutôt *ces* orbites périodiques, car il fit varier le seul paramètre qui figurât dans ces équations, le paramètre m); il détermina point par point ces trajectoires fermées et calcula les coordonnées de ces points avec de nombreuses décimales. Les développements de Delaunay furent remplacés par d'autres plus convergents et pour de grandes valeurs de m, quand les séries nouvelles elles-même ne suffirent plus, M. Hill eut recours aux quadratures mécaniques. Il arrive finalement au cas, où, pour l'observateur mobile dont nous parlions, l'orbite apparente aurait un point de rebroussement.

Une dernière remarque; M. Hill, dans le mémoire que nous analysons, transforme ses équations de façon à les rendre homogènes et il tire de ces équations homogènes un remarquable parti; il serait aisé de faire quelque chose d'analogue dans le cas général du problème des trois corps; il suffirait d'éliminer les masses entre les équations du mouvement; l'ordre de ces équations se trouverait ainsi augmenté, mais on arriverait à n'avoir plus dans les deux membres que des polynômes entiers par rapport aux coordonnées rectangulaires et à leurs dérivées. Les équations ainsi obtenues ne pourraient servir à l'intégration, mais elles pourraient rendre de précieux services comme formules de vérification.

Par ce mémoire les termes qui ne dépendent que de m se trouveraient entièrement déterminés avec une précision infiniment plus grande que dans aucune des théories antérieures; les termes les plus importants ensuite sont

ceux qui sont proportionnels à l'excentricité de la Lune et ne dépendent d'ailleurs que de m. Ces termes dépendent des mêmes équations différentielles; mais comme on connaît déjà une solution de ces équations et que celle que l'on cherche en diffère infiniment peu, tout se ramène à la considération des "équations aux variations." Or ces équations sont linéaires, elles sont à coëfficients périodiques; elles sont du 4ème ordre, mais la connaissance de l'intégrale de Jacobi permet de les ramener aisément au 2ème ordre. La théorie générale des équations linéaires à coëfficients périodiques nous apprend que ces équations admettent deux solutions particulières susceptibles d'être représentées par une fonction périodique multipliée par une exponentielle. C'est l'exposant de cette exponentielle qu'il s'agit d'abord de déterminer et cet exposant a une signification physique très simple et très importante, puisqu'il représente le moyen mouvement du périgée.

La solution adoptée par M. Hill est aussi originale que hardie. Notre équation différentielle doit être résolue par une série. En y substituant une série S à coëfficients indéterminés, on obtiendra une autre série Σ qui devra être identiquement nulle. En égalant à zéro les différents coëfficients de cette série Σ, on obtiendra des équations linéaires où les inconnues seront les coëfficients indéterminés de la série S. *Seulement ces équations de même que les inconnues étaient en nombre infini.* Avait-on le droit d'égaler à zéro le déterminant de ces équations? M. Hill l'a osé et c'était là une grande hardiesse; on n'avait jamais jusque-là considéré des équations linéaires en nombre infini; on n'avait jamais étudié les déterminants d'ordre infini; on ne savait même pas les définir et on n'était pas certain qu'il fût possible de donner à cette notion un sens précis. Je dois dire cependant, pour être complet, que M. Kotteritzsch avait dans les Poggendorf's Annales abordé le sujet. Mais son mémoire n'était guère connu dans le monde scientifique et en tout cas ne l'était pas de M. Hill. Sa méthode n'a d'ailleurs rien de commun avec celle du géomètre américain.

Mais il ne suffit pas d'être hardi, il faut que la hardiesse soit justifiée par le succès. M. Hill évita heureusement tous les piéges dont il était environné, et qu'on ne dise pas qu'en opérant de la sorte il s'exposait aux erreurs les plus grossières; non, si la méthode n'avait pas été légitime, il en aurait été tout de suite averti, car il serait arrivé à un résultat numérique absolument différent de ce que donnent les observations. La même méthode donne les coëfficients des diverses inégalités proportionnelles à l'excentricité et dont les plus importantes sont l'équation du centre et l'évection. Comparons ce calcul avec celui de Delaunay; la méthode de Hill avec deux ou trois approximations donne un grand nombre de décimales; Delaunay pour en avoir moitié moins devait prendre huit termes dans sa série, et ce n'était pas

assez, il fallait évaluer par des procédés approchés le *reste* de la série ; s'il avait fallu attendre qu'on arrive à des termes négligeables, la plus robuste patience se serait lassée. A quoi tient cette différence? Le mouvement g du périgée nous est donné par la formule

$$\cos g\pi = \varphi(m)$$

$\phi(m)$ étant une série procédant suivant les puissances de m et rapidement convergente. M. Hill calcule directement $\cos g\pi$ et en déduit facilement g.

Au contraire, Delaunay s'efforce de développer g suivant les puissances croissantes de m. Or, la convergence du développement est beaucoup plus lente. On ne doit pas s'en étonner, si l'on supposait par exemple

$$\cos g\pi = 1 - \alpha m$$

on aurait $\cos g\pi$ tout de suite, tandis que le développement de g suivant les puissances de m convergerait très lentement si αm était très voisin de 1 et ne convergerait plus du tout si αm était plus grand que 1.

Et ce n'est pas tout, Delaunay traîne désormais un boulet dont il ne peut se débarrasser et qui dans toute la suite de ses calculs s'oppose à la convergence rapide de ses séries. Il serait amené à des séries de la forme $\Sigma A_n m^n$ dont les termes décroîtraient assez vite. Mais les coëfficients A_n dépendent de g, et g dépend de m. Comme il veut tout développer suivant les puissances de m, il développe ces coëfficients A_n. Or, le développement de A_n, et par conséquent les séries finales ne peuvent converger plus vite que g; nous sommes donc condamnés à n'avoir plus que des convergences très lentes.

On voit par ces considérations toute l'étendue du progrès réalisé par M. Hill. La méthode qui avait réussi pour le mouvement du périgée pouvait être appliquée au mouvement du nœud.

Désormais, les principales difficultés sont vaincues et les approximations suivantes sont plus aisées; les termes dépendant de l'excentricité ou de la parallaxe solaire, ou bien des puissances supérieures de l'excentricité lunaire, peuvent se calculer plus facilement; on n'a plus qu'à intégrer des équations linéaires à second membre, sachant intégrer les équations sans second membre, puisque ces équations sans second membre ne sont pas autre chose que celles même que M. Hill a eu à résoudre pour trouver le mouvement du périgée.

La méthode classique de la variation des constantes donne immédiatement la solution. On rencontre, néanmoins, encore des difficultés pratiques. M. Hill en signale une dans un article de l'Astronomical Journal, No. 471 (On the Inequalities in the Lunar Theory strictly proportional to the Solar Eccentricity). En dirigeant d'une certaine manière le calcul on arrive très vite à exprimer la solution par deux quadratures; mais les fonctions sous le

signe $\int$ ne sont pas développables en séries trigonométriques, car elles sont susceptibles de devenir infinies. Pour éviter cette difficulté, M. Hill revient aux coordonnées polaires. Ce n'est pas là la solution qu'a adoptée dans ces derniers temps M. Brown; celle-ci est plus satisfaisante à beaucoup d'égards que celle de M. Hill; nous devons toutefois faire observer qu'elle oblige à quatre quadratures et que chacun des quatre termes ainsi obtenus est beaucoup plus grand en valeur absolue que le chiffre qui exprime le résultat final du calcul, c'est à dire que la somme algébrique des quatre termes.

C'est sur ces principes qu'est fondée la nouvelle théorie de M. Brown. Celle-ci est beaucoup plus parfaite que toutes les théories de la Lune que nous connaissons et il y a lieu d'espérer qu'elle permettra de pousser l'approximation plus loin que ne l'avaient fait Hansen et Delaunay. Il serait injuste de méconnaître la part personnelle que M. Brown a prise à ce grand travail, et l'originalité des idées qui lui appartiennent en propre. Mais il serait plus injuste encore d'oublier que c'est M. Hill qui a posé les principes; qu'il a vaincu les premières difficultés et que ces difficultés étaient les plus grandes. La nouvelle théorie tient le milieu entre celle de Hansen et celle de Delaunay, elle n'est ni purement numérique comme la première, ni purement littérale comme la seconde; la lettre m est seule remplacée par sa valeur numérique; les lettres qui désignent les autres constantes continuent à figurer explicitement.

Dans la théorie de la Lune, il convient de faire deux parts; il faut étudier d'abord les inégalités dues à l'action du Soleil; ce sont celles qui se produiraient si la Terre, le Soleil et la Lune existaient seuls et se réduisaient à des points matériels. Nous venons de voir ce que M. Hill et M. Brown nous en ont fait connaître. Mais ces inégalités ne sont pas les seules. En dehors du Soleil et de la Lune, il y a les planètes qui troublent le mouvement de notre satellite, d'abord par leur action directe, et ensuite parce que, par suite de leur attraction, le mouvement relatif de la Terre et du Soleil ne suit plus les lois de Képler. D'autre part la Terre n'est pas sphérique, et la Lune en est si rapprochée que l'attraction du renflement équatorial influe sur son mouvement.

Dans l'effet des planètes nous devons distinguer les variations séculaires, les plus importantes et les plus délicates de toutes. M. Hill s'en est occupé à diverses reprises; il a étudié successivement l'accélération séculaire du moyen mouvement, celle du mouvement du périgée et l'influence des variations de l'écliptique. Nous avons d'autre part les inégalités planétaires périodiques et surtout celles dont la période est assez longue; ce sont celles-là qui nous donnent encore aujourd'hui le plus de soucis, car on n'est jamais sûr de n'en

avoir pas oublié. M. Neison en avait découvert une nouvelle due à Jupiter; M. Hill a montré qu'il s'était trompé dans le calcul du coëfficient; on lira cette discussion avec le plus grand intérêt.

Enfin, il a consacré un assez long mémoire à l'influence de l'aplatissement terrestre; nous signalerons surtout la discussion des observations de pendule, faite en déterminant les coëfficients numériques des différentes inégalités. Le théorème de Stokes nous apprend, en effet, que l'attraction du sphéroïde terrestre sur un point extérieur, et en particulier sur la Lune, est entièrement déterminée quand on connaît l'intensité de la pesanteur en tous les points de la surface terrestre.

La théorie de la Lune n'absorbait pas cependant toute son activité et les perturbations des planètes attirèrent également son attention; la question classique du développement de la fonction perturbatrice et les généralités sur le problème des 3 corps, la théorie de Céres et d'Hestia sont l'objet de plusieurs mémoires, mais nous nous arrêterons surtout sur un ouvrage de longue haleine, dont l'importante pratique est très grande.

La théorie de Jupiter, et en particulier la détermination de la masse de cette planète, l'avaient déjà occupé à plusieurs reprises quand il aborda l'étude complète des perturbations mutuelles de Jupiter et de Saturne.

Laplace avait abordé cette théorie, qui présente de grandes difficultés à cause de la grande inégalité, mais ses évaluations des termes du 2d ordre n'étaient que grossièrement approchées. Hansen fut plus heureux et dirigea le calcul de façon qu'il soit aisé de se rendre compte de l'importance des termes négligés, mais il n'a traité complètement que le cas de Saturne, se bornant pour Jupiter aux termes du premier ordre.

Les mémoires qui suivirent jusqu'à celui de Le Verrier ont peu ajouté à nos connaissances sur le sujet; mais en 1876 Le Verrier publia une théorie tout à fait complète; ses formules sont entièrement littérales, de sorte que si l'on est amené à apporter de petites corrections aux éléments, on trouvera immédiatement les corrections correspondantes des coëfficients des inégalités. D'ailleurs ce ne sont pas les coordonnées qui sont calculées mais les éléments elliptiques osculateurs, conformément à l'esprit de la méthode de la variation des constantes.

Cette façon de procéder avait ses avantages, mais elle exigeait un surcroît considérable de labeur, et comme résultat final la précision est insuffisante, de sorte que pour une partie de son calcul Le Verrier est forcé d'en revenir aux quadratures mécaniques. Les tables de Le Verrier ne sont d'ailleurs pas d'un usage commode. Mais ce n'est pas pour ces raisons que M. Hill entreprit son travail; au moment où il commença les tables de Le Verrier n'étaient pas publiées et on ne savait trop quand elles le seraient.

Celles qui étaient en usage, c'est à dire celles de Bouvard, ne répondaient plus aux besoins de l'Astronomie. Le but poursuivi par l'auteur était purement pratique, il fallait obtenir de bonnes tables dans un temps très court. C'est pourquoi il ne voulut pas perdre de temps à chercher une méthode nouvelle, et il se contenta de celle de Hansen. Nous ne devons donc pas nous attendre à trouver dans ce nouvel ouvrage la même originalité que dans les études sur la Lune. Les perfectionnements apportés à la méthode de Hansen ne porteront que sur des détails; ce sont avant tout des simplifications; c'est ainsi, par exemple, que pour ne pas avoir deux variables indépendantes, il ne fait pas de l'anomalie excentrique le même usage que Hansen; et cet usage en effet n'est justifié que si l'on doit se borner aux termes du premier ordre. M. Hill évite ainsi plusieurs transformations de séries qui allongeaient inutilement le travail dans la forme primitive de la méthode de Hansen.

Un autre perfectionnement consiste à incorporer parmi les termes du 2d ordre,. les plus importants des termes du 3e ordre. Remarquons que c'était là se rapprocher de la méthode de Delaunay, qui serait à mon sens celle qu'il conviendrait d'employer pour l'étude de l'action mutuelle de Jupiter et de Saturne.

Une question délicate était celle du choix des valeurs à attribuer aux masses. M. Hill a été conduit à modifier les masses adoptées par Le Verrier, et c'est là peut-être la principale cause des divergences que l'on remarque entre ses tables et celles de son devancier.

Le résultat de ce long travail a été un volume de recherches théoriques et deux volumes de tables précises et commodes; l'un pour le mouvement de Jupiter, l'autre pour celui de Saturne.

Les derniers progrès de la Mécanique Céleste attiraient constamment l'attention de M. Hill, qui cherchait à s'assimiler et à éprouver les méthodes récemment proposées; nous venons de voir comment il avait transformé et appliqué à Jupiter la méthode de Hansen; il a publié d'autre part sur cette même méthode une étude critique dans l' "American Journal of Mathematics."

De même il ne pouvait manquer de soumettre à la discussion les travaux si intéressants de Gyldén; c'était l'époque où l'astronome suédois introduisait dans la Science la notion d'orbite intermédiaire. Cette idée de substituer à l'ellipse Képlerienne une orbite plus approchée était trop ingénieuse pour ne pas le frapper. Non seulement il a analysé et discuté les principaux mémoires de Gyldén, mais il a lui-même proposé une orbite intermédiaire qui pourrait être employée avec avantage dans la théorie de la Lune. On n'a qu'à distraire de la fonction perturbatrice deux termes destinés à mettre en évidence le mouvement du périgée et celui du nœud, et à tenir compte

de ces deux termes dès la première approximation. Il y a là une idée dont on aurait pu tirer parti, si M. Hill ne l'avait lui-même d'avance rendue inutile par la perfection de ses premiers travaux.

Il comprit également la portée de la méthode de Delaunay; dans plusieurs notes il a montré que cette méthode n'est pas limitée à la théorie de la Lune et qu'on peut l'employer utilement dans le calcul des perturbations planétaires. Certes ce résultat n'était pas nouveau et Tisserand l'avait établi depuis longtemps, mais M. Hill a beaucoup ajouté à nos connaissances à ce sujet en approfondissant les conditions dans lesquelles ces nouveaux procédés sont applicables au calcul du mouvement d'Hécube ou des variations séculaires des excentricités et des inclinaisons. L'étude du mouvement d'Hécube se rattachait d'ailleurs naturellement pour lui à la recherche de ces solutions périodiques, à la découverte desquelles il avait eu une si grande part.

Il s'occupa rarement de la théorie de la figure de la Terre et de celle de la précession. Néanmoins un de ses premiers articles est relatif à la première de ces questions et il y est revenu plus récemment à deux reprises.

Ainsi aucune des parties de la Mécanique Céleste ne lui a été étrangère, mais son œuvre propre, celle qui fera son nom immortel, c'est sa théorie de Lune; c'est là qu'il a été non seulement un artiste habile, un chercheur curieux, mais un inventeur original et profond. Je ne veux pas dire que ces méthodes qu'il a créées ne sont applicables qu'à la Lune; je suis bien persuadé du contraire, je crois que ceux qui s'occupent des petites planètes seront étonnés des facilités qu'ils rencontreront le jour où en ayant pénétré l'esprit ils les appliqueront à ce nouvel objet. Mais jusqu'ici c'est pour la Lune qu'elles ont fait leurs preuves; quand elles s'étendront à un domaine plus vaste, on ne devra pas oublier que c'est à M. Hill que nous devons un instrument si précieux.

Mars 1905.

THE COLLECTED
MATHEMATICAL WORKS
OF
GEORGE WILLIAM HILL

Ἄστρων κάτοιδα νυκτέρων ὁμήγυριν.—Æschylus.

THE

COLLECTED MATHEMATICAL WORKS

OF

G. W. HILL

MEMOIR No. 1.

On the Curve of a Drawbridge.

(Runkle's Mathematical Monthly, Vol. I, pp. 174–175, 1859.)

In Mr. Watson's article in the January number of the monthly, On the Curve of a Drawbridge, I would like to notice that the investigation could be much shortened. For, drawing vertical lines from the roller E and center of gravity of the platform, along which lines the weights W_1 and W tend to move, and applying the principle of virtual velocities, we have

$$W_1\delta\,(a - r\cos\varphi) - W\delta\,(\tfrac{1}{2}\,l\cos\theta) = 0\,;$$

or

$$W_1\delta\,(r\cos\varphi) + W\delta\left(\frac{l\,[(c-r)^2 - 2a^2]}{4a^2}\right) = 0\,.$$

By integrating

$$W_1 r\cos\varphi + W\frac{l\,[(c-r)^2 - 2a^2]}{4a^2} = \text{constant}$$

or since

$$\frac{2\,W_1 a^2}{Wl} = B,$$

$$2Br\cos\varphi - 2cr + r^2 = \text{constant}$$

which is the equation to the curve.

Rutgers College, January 22, 1859.

MEMOIR No. 2.

Discussion of the Equations which Determine the Position of a Comet or other Planetary Body from Three Observations.

(Runkle's Mathematical Monthly, Vol. III, pp. 26–29, 1860.)

These equations may be found in the *Mécanique Céleste*, Tom. I, p. 207, or in Pontécoulant, *Théorie Analytique du Système du Monde*, Tom. II, p. 21, where the analysis conducting to them is given. The equations are

$$\rho = \frac{R^4}{m}\left(\frac{1}{r^3} - \frac{1}{R^3}\right), \quad r^2 = R^2 - 2R\rho \cos c + \rho^2 ;$$

in which ρ is the comet's distance from the earth, r its radius vector, R the earth's radius vector, c the angular distance of the comet from the sun, and m a function of known quantities. The unknown quantities are ρ and r. Now, to find the values of ρ and r, it has always been advised to determine them from the equations in the above form by the method of double position. But this way is needlessly complex in calculation, and we shall give the equations a much simpler form, to which the tentative process can be easily applied.

Assume

$$r = \frac{R \sin c}{\sin \theta} \text{ and } \rho = \frac{R \sin (\theta - c)}{\sin \theta} ;$$

these values, substituted in the last equation, render it identical; and from the first we obtain

$$\frac{\sin^3 \theta}{\sin^3 c} = 1 + m \frac{\sin (\theta - c)}{\sin \theta} ,$$

or,

$$\sin^4 \theta = \sin^3 c \left[(1 + m \cos c) \sin \theta - m \sin c \cos \theta\right] .$$

Let

$$\sin^3 c (1 + m \cos c) = A \cos \beta , \quad m \sin^4 c = A \sin \beta ;$$

whence

$$\tan \beta = \frac{m \sin c}{1 + m \cos c} , \quad A = \frac{m \sin^4 c}{\sin \beta} = \frac{\sin^4 c}{\sin (c - \beta)} .$$

Then the equation becomes

$$\sin^4 \theta = A \sin (\theta - \beta) .$$

This is probably the most elegant form to which the above equations can be reduced. Taking the logarithms of both sides, we have

$$4 \log \sin \theta - \log \sin (\theta - \beta) = \log A .$$

From this a near approximate value of θ can be found by inspection of the table of logarithmic sines, without any calculation but that which can be performed mentally. The exact value may then be obtained by a single application of the method of double position.

We proceed to notice some properties of the roots of the equation to which we have reduced the two first given.

The equation may be put under this form

$$\sin^8\theta - 2A\cos\beta\sin^5\theta + A^2\sin^2\theta - A^2\sin^2\beta = 0.$$

From the powers of $\sin\theta$, which are wanting, we perceive that the equation has at least four imaginary roots; and the sign of the last term being negative, there are at least two real roots, one positive, the other negative. By reference to the equations which determine ρ and r, it will be seen that they are satisfied by the values $\rho = 0$ and $r = R$; the corresponding value of θ is c, which, put in the equation $\sin^4\theta = A\sin(\theta - \beta)$, renders it identical. Since, by the nature of the problem, only positive values of ρ and r are admissible, and consequently θ being contained between the limits c and π, we may reject as useless the four imaginary roots, the negative root and the root $\theta = c$. There remain two roots which are necessarily real, because the problem must have at least one solution, and we are led to the important conclusion that it cannot have more than two.

Let us here deduce a relation which exists between the known quantities when these two roots are real. Differentiating the equation

$$4\log\sin\theta - \log\sin(\theta - \beta) = \log A$$

with respect to θ, we get for the equation containing the values of θ, when the left member is a maximum or minimum,

$$4\cot\theta - \cot(\theta - \beta) = 0, \quad \text{or} \quad 4\tan(\theta - \beta) = \tan\theta;$$

from which we derive

$$\tan\theta = \frac{3 \pm \sqrt{(9 - 16\tan^2\beta)}}{2\tan\beta}.$$

If $\tan^2\beta > \frac{9}{16}$, this value of $\tan\theta$ is imaginary, and the left member of the equation differentiated is not susceptible of a maximum or minimum value, and the equation in $\sin\theta$ has only two real roots, which are among those rejected. Hence we conclude, that when the data are taken from observation, the quantity $\dfrac{m\sin c}{1 + m\cos c}$ will always be contained between the

limits $\pm \frac{3}{4}$. If we substitute 0 for θ in the equation $\sin^4 \theta - A \sin (\theta - \beta) = 0$, the result is $A \sin \beta$, and for $\theta = \pi$, the result is $-A \sin \beta$; showing the existence of and odd number of roots between the limits $\theta = 0$ and $\theta = \pi$, which odd number is three, since c and the root which the problem demands are within these limits. If we make $\theta = c + dc$, there results the quantity

$$[4 \sin^3 c \cos c - A \cos (c - \beta)] \, dc ;$$

or, since

$$A = \frac{\sin^4 c}{\sin (c - \beta)}, \quad \tan \beta = \frac{m \sin c}{1 + m \cos c},$$

the quantity

$$(3 \cos c - m) \sin^3 c dc.$$

And $-A \cos \beta$, the result, on putting $\theta = \pi$, is equal to $-m \sin^4 c$. Therefore, $\sin c$ being always positive, θ has two real values, or only one (between the limits c and π), and, consequently, the problem two or one answer, according as m and $m - 3 \cos c$ have the same or opposite signs.

It is evident that $A \cos \beta$ must be positive, in order that the equation in $\sin \theta$ may have three positive real roots; so the quantity $1 + m \cos c$ is always positive, and $\tan \beta$ has the same sign as m. If β be taken between the limits $\pm \frac{\pi}{2}$, A is always positive. Since the equation in $\sin \theta$ must have no root greater than one, unity substituted for $\sin \theta$ in the first derived function of its equation must render it positive; that is, the expression $4 - 5A \cos \beta + A^2$ is positive, which gives $A < 2$ and $A \cos \beta < \frac{8}{5}$. According as m is positive or negative, the equation for finding θ presents itself under two shapes, $\sin^4 \theta = A \sin (\theta - \beta)$ and $\sin^4 \theta = A \sin (\theta + \beta)$, in which A is always positive and less than 2, and β never exceeds $36° \, 53'$.

From the expression for ρ in terms of r, it is clear that r is less or greater than R, according as m is positive or negative. Therefore, in the first case, θ is contained between the limits c and $\pi - c$; and, in the second case, if c is in the first quadrant, between $\pi - c$ and $\pi - \beta$; but if c be in the second quadrant between c and $\pi - \beta$. These remarks may be of use to shorten the tentative process of finding θ.

With regard to θ, it is clear it is the angle subtended at the comet by its radius vector and the line joining it and the earth prolonged beyond the comet.

MEMOIR No. 3.

On the Conformation of the Earth.*

(First Prize Essay, Runkle's Mathematical Monthly, Vol. III, pp. 166–182, 1861.)

1. All the particles which compose the mass of the earth are animated by the attraction of gravitation. The law of this force is, that the attraction of any atom for a spherical surface of material points, described about it as a center, is constant. Hence, if the attraction of an atom for a material point be represented by A, and r be the radius of the spherical surface and N the number of material points in a unit of surface, the attraction of the central atom for the spherical surface is $4\pi Nr^2A =$ a constant $= -4\pi NM$. Whence $A = -\frac{M}{r^2}$; that is, the attraction varies inversely as the distance squared. The constant M is called the mass of the attracting atom. We have given A the negative sign because it represents a force tending to decrease the line r.

2. Making $A = \frac{\partial V}{\partial r}$, then $V = \frac{M}{r}$. V is called the *potential function*, and has this property: that if the partial derivative of it be taken with respect to any of the three spaceal coordinates of which it is necessarily a function, the result will be the partial force in the direction of that coordinate axis.

3. If the attraction of a single atom give $V = \frac{M}{D}$, D denoting the distance, the attraction of an indefinite number or assemblage of atoms will give $V = S.\frac{M}{D}$. If ξ, ψ, ϕ represent any three lines at right angles with each other, then $\frac{\partial V}{\partial \xi}$, $\frac{\partial V}{\partial \psi}$, $\frac{\partial V}{\partial \phi}$ are the forces acting in each of these directions respectively. In a rectangular system

$$D = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}},$$

*This memoir, written at the end of 1859 and beginning of 1860, was designed to show how all the formulae connected with the figure of the earth could be derived from Laplace's and Poisson's equations, combined with the hydrostatic equilibrium of the surface, without any appeal to the definite integrals belonging to the subject of attraction of spheroids. Some of the assumptions are quite unwarranted, nevertheless I allow them to stand.

where the accented letters pertain to the attracting atom, and the unaccented to the attracted. Consequently,

$$\frac{\partial V}{\partial x} = S.\frac{M}{D^3}(x'-x),\quad \frac{\partial V}{\partial y} = S.\frac{M}{D^3}(y'-y),\quad \frac{\partial V}{\partial z} = S.\frac{M}{D^3}(z'-z). \tag{1}$$

4. Differentiating again,

$$\left.\begin{aligned}\frac{\partial^2 V}{\partial x^2} &= S.\frac{M}{D^3}\left\{\frac{3}{D^2}(x'-x)^2-1\right\},\\ \frac{\partial^2 V}{\partial y^2} &= S.\frac{M}{D^3}\left\{\frac{3}{D^2}(y'-y)^2-1\right\},\\ \frac{\partial^2 V}{\partial z^2} &= S.\frac{M}{D^3}\left\{\frac{3}{D^2}(z'-z)^2-1\right\}.\end{aligned}\right\} \tag{2}$$

In this differentiation M has been regarded as independent of x, y, z; but, in order to render equations (2) altogether general, the attracting mass must be considered as extending into the point x, y, z. Let ρ be the density of the atom occupying this point, which becomes zero when the attracting mass does not reach the point. This atom may be regarded as spherical, then for it $M = \frac{4}{3}\pi\rho D^3$; substituting this value in equations (1), the results are

$$\frac{\partial V}{\partial x} = \tfrac{4}{3}\pi\rho(x'-x),\quad \frac{\partial V}{\partial y} = \tfrac{4}{3}\pi\rho(y'-y),\quad \frac{\partial V}{\partial z} = \tfrac{4}{3}\pi\rho(z'-z).$$

Hence, we must add the term $-\frac{4}{3}\pi\rho$ to the right members of equations (2), and then we can regard D as having always a finite value. By adding these equations, there results

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + 4\pi\rho = 0. \tag{3}$$

The integration of this gives V; ρ is a function of x, y, z; in the case of solid bodies, as the earth, a limited function.

5. Transform (3) to terms of polar coordinates; put

$$x = r\sqrt{(1-\mu^2)}\cos\omega,\quad r = \sqrt{(x^2+y^2+z^2)},$$

$$y = r\sqrt{(1-\mu^2)}\sin\omega,\quad \mu = \frac{z}{\sqrt{(x^2+y^2+z^2)}},$$

$$z = r\mu,\qquad \omega = \tan^{-1}\frac{y}{x}.$$

Then

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\left\{\frac{\partial r^2}{\partial x^2}+\frac{\partial r^2}{\partial y^2}+\frac{\partial r^2}{\partial z^2}\right\}\frac{\partial^2 V}{\partial r^2}+\left\{\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right\}\frac{\partial V}{\partial r}$$

$$+\left\{\frac{\partial \mu^2}{\partial x^2}+\frac{\partial \mu^2}{\partial y^2}+\frac{\partial \mu^2}{\partial z^2}\right\}\frac{\partial^2 V}{\partial \mu^2}+\left\{\frac{\partial^2 \mu}{\partial x^2}+\frac{\partial^2 \mu}{\partial y^2}+\frac{\partial^2 \mu}{\partial z^2}\right\}\frac{\partial V}{\partial \mu}$$

$$+\left\{\frac{\partial \omega^2}{\partial x^2}+\frac{\partial \omega^2}{\partial y^2}+\frac{\partial \omega^2}{\partial z^2}\right\}\frac{\partial^2 V}{\partial \omega^2}+\left\{\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}+\frac{\partial^2 \omega}{\partial z^2}\right\}\frac{\partial V}{\partial \omega}$$

$$+2\left\{\frac{\partial r}{\partial x}\frac{\partial \mu}{\partial x}+\frac{\partial r}{\partial y}\frac{\partial \mu}{\partial y}+\frac{\partial r}{\partial z}\frac{\partial \mu}{\partial z}\right\}\frac{\partial^2 V}{\partial r\partial \mu}$$

$$+2\left\{\frac{\partial \mu}{\partial x}\frac{\partial \omega}{\partial x}+\frac{\partial \mu}{\partial y}\frac{\partial \omega}{\partial y}+\frac{\partial \mu}{\partial z}\frac{\partial \omega}{\partial z}\right\}\frac{\partial^2 V}{\partial \mu\partial \omega}$$

$$+2\left\{\frac{\partial \omega}{\partial x}\frac{\partial r}{\partial x}+\frac{\partial \omega}{\partial y}\frac{\partial r}{\partial y}+\frac{\partial \omega}{\partial z}\frac{\partial r}{\partial z}\right\}\frac{\partial^2 V}{\partial \omega\partial r}$$

$$=r^{-2}\left\{\frac{\partial\,.\,r^2\frac{\partial V}{\partial r}}{\partial r}+\frac{\partial\,.\,(1-\mu^2)\frac{\partial V}{\partial \mu}}{\partial \mu}+\frac{\frac{\partial^2 V}{\partial \omega^2}}{1-\mu^2}\right\}.$$

Hence (3) becomes

$$\frac{\partial\,.\,r^2\frac{\partial V}{\partial r}}{\partial r}+\frac{\partial\,.\,(1-\mu^2)\frac{\partial V}{\partial \mu}}{\partial \mu}+\frac{\frac{\partial^2 V}{\partial \omega^2}}{1-\mu^2}+4\pi\rho r^2=0. \tag{4}$$

6. To show the application of (4), take the simple case when the surfaces of equal density are concentrically spherical. Placing the origin of coordinates at the common centre, ρ becomes a function of r alone, either continuous or discontinuous as the case demands; and evidently $\frac{\partial V}{\partial \mu}=0$, $\frac{\partial V}{\partial \omega}=0$; therefore, (4) becomes

$$\frac{\partial\,.\,r^2\frac{\partial V}{\partial r}}{\partial r}+4\pi\rho r^2=0.$$

By integration

$$\frac{\partial V}{\partial r}=\left(C-4\pi\int\rho r^2dr\right)r^{-2}.$$

Between the limits 0 and r, $4\pi\int\rho r^2 dr$ is equal to the mass contained within the sphere whose radius is r; denoting this by M, it is clear that $C=0$, since the expression must agree with that for the attraction of a single atom. Thus $\frac{\partial V}{\partial r}=-\frac{M}{r^2}$. Or the principle may be stated: *The force acting on any point, wherever situated, equals the mass of all the particles nearer the center than the point attracted, divided by the square of the distance of the point from that center, taken with the negative sign.*

7. The earth revolves about a constant axis; hence, to remove our problem from dynamics to statics, it is necessary to introduce the force of pressure called the centrifugal force. Making the coordinal axis of z coincide with the earth's axis, and T denoting the period of the earth's rotation, the potential of the centrifugal force is

$$V = \frac{2\pi^2}{T^2}(x^2 + y^2) = \frac{2\pi^2}{T^2} r^2 (1 - \mu^2).$$

Since this force animates every particle, include its potential in V and make V the potential of both gravitating and centrifugal force. It then becomes necessary to add to (4) the term $-\frac{8\pi^2}{T^2} r^2$, whence

$$\frac{\partial . r^2 \frac{\partial V}{\partial r}}{\partial r} + \frac{\partial . (1 - \mu^2) \frac{\partial V}{\partial \mu}}{\partial \mu} + \frac{\frac{\partial^2 V}{\partial \omega^2}}{1 - \mu^2} + 4\pi \left(\rho - \frac{2\pi}{T^2}\right) r^2 = 0. \qquad (5)$$

8. To apply this equation to the solution of the earth's conformation, we must combine it with some condition of equilibrium. From the manner in which the atmosphere and ocean cover the earth, we may conjecture it was once fluid, and in solidifying, preserved the form it had taken by the laws of hydrostatics. In passing from the solid earth to the ocean, and from the ocean to the atmosphere, there occur two faults in the continuity of the earth's density; hence ρ is strictly represented by a discontinuous function. But, as the mass of the ocean and atmosphere is about $\frac{1}{4000}$ of the whole, its influence may be neglected, and ρ supposed continuous from center to surface.

9. If p is the pressure, then $dp = \rho d V$, and $V + B = 0$ is the equation to surfaces of level, B having a different value for each surface. Let ρ be a function of p, and thus of V. In (5) make

$$4\pi \left(\rho - \frac{2\pi}{T^2}\right) = f(V).$$

10. The centrifugal force being small compared with gravity, may be regarded as a perturbing force. Supposing at first this force is zero, the particles would arrange themselves symmetrically about a center, since there is no reason why they should accumulate more in one place than in another. Take the origin of coordinates at this center, then $\frac{\partial V}{\partial \mu} = 0$, $\frac{\partial V}{\partial \omega} = 0$. Thus (5) becomes

$$\frac{\partial . r^2 \frac{\partial V}{\partial r}}{\partial r} + r^2 f(V) = 0.$$

Consequently V is a function of r alone, and the general equation of surfaces of level $V + B = 0$, when solved gives $r =$ a constant; these surfaces are then concentrically spherical.

11. The term $\frac{2\pi^2}{T^2} r^2 (1 - \mu^2)$, which the centrifugal force introduces into V, and which causes a departure from the spherical form, does not contain ω, and so whatever derangement it may produce, cannot introduce ω into V; that is, the earth is a solid of revolution. Consequently (5) becomes

$$\frac{\partial . r^2 \frac{\partial V}{\partial r}}{\partial r} + \frac{\partial . (1-\mu^2)\frac{\partial V}{\partial \mu}}{\partial \mu} + r^2 f(V) = 0. \tag{6}$$

12. From the form of this same term, it may be concluded that

$$V = Y_0 + Y_1\mu^2 + Y_2\mu^4 + \ldots = \Sigma . Y_i \mu^{2i}, \tag{7}$$

where Y_i is a function of r alone, and a quantity of the order of the i^{th} power of the centrifugal force. Substitute this expression of V in (6) and put the coefficient of μ^{2i} resulting equal to zero, and let the coefficient of μ^{2i} in $r^2 f(V)$ be U_i; then

$$\frac{d . r^2 \frac{dY_i}{dr}}{dr} + (2i+1)(2i+2)\, Y_{i+1} - 2i(2i+1)\, Y_i + U_i = 0. \tag{8}$$

This equation has the inconvenience of introducing Y_{i+1}; let us therefore assume more generally $V = \Sigma . V_i M_i$, V_i being a function of r of the same order as Y_i, and M_i a function of μ. Making these substitutions in (6),

$$\Sigma . \frac{d . r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma . V_i \frac{d . (1-\mu^2)\frac{dM_i}{d\mu}}{d\mu} + r^2 f(\Sigma . V_i M_i) = 0,$$

which may be written

$$\Sigma \frac{d . r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma . V_i \frac{d . (1-\mu^2)\frac{dM_i}{d\mu}}{d\mu} + \Sigma . T_i M_i = 0. \tag{9}$$

In order that the left member may be arranged in a series of the same form as $\Sigma \,.\, V_i M_i$, we must have

$$\frac{d \,.\, (1-\mu^2)\dfrac{dM}{d\mu}}{d\mu} = nM_i,\text{*}$$

in which n is independent of μ. We may determine n from the consideration that, V_i being of the same order as Y_i, M_i cannot contain any higher power of μ than μ^{2i}. Making $M_i = \Sigma \,.\, k_s \mu^{2s}$, this relation results:

$$k_{s+1} = \frac{n + 2s\,(s+1)}{(2s+1)(2s+2)} k_s \,.$$

To make this series end at k_i, n must equal $-\,2i\,(2i+1)$; and

$$k_{s+1} = -\frac{(2i-2s)(2i+2s+1)}{(2s+1)(2s+2)} k_s \,;$$

hence, putting $k_0 = 1$, which is allowable,

$$M_i = 1 - \frac{2i\,(2i+1)}{1\,.\,2}\mu^2 + \frac{(2i-2)\,2i\,(2i+1)(2i+3)}{1\,.\,2\,.\,3\,.\,4}\mu^4 \ldots \pm \frac{2\,.\,.\,.\,2i\,(2i+1)\,.\,.\,.\,(4i+1)}{1\,.\,2\,.\,3\,.\,.\,.\,2i}\mu^{2i} \,. \tag{10}$$

For V_i we have from (9), by rejecting the sign Σ and dividing by M_i, the equation

$$\frac{d \,.\, r^2 \dfrac{dV_i}{dr}}{dr} - 2i\,(2i+1)\; V_i + T_i = 0 \,. \tag{11}$$

From the expression (10) we easily deduce

$$Y = \pm \frac{(2i+1)(2i+3)\,.\,.\,.\,(4i-1)}{1\,.\,3\,.\,.\,.\,(2i-1)} \left\{ V_i + \frac{(i+1)(4i+1)}{1\,.\,(2i+1)} V_{i+1} + \frac{(i+1)(i+2)(4i+1)(4i+3)}{1\,.\,2\,.\,(2i+1)(2i+3)} V_{i+2} + \ldots \right\}. \tag{12}$$

The inversion of which is

$$V_i = \pm \frac{1\,.\,3\,.\,.\,.\,(2i-1)}{(2i+1)(2i+3)\,.\,.\,.\,(4i-1)} \left\{ Y_i + \frac{(i+1)(2i+1)}{1\,.\,(4i+3)} Y_{i+1} + \frac{(i+1)(i+2)(2i+1)(2i+3)}{1\,.\,2\,.\,(4i+3)(4i+5)} Y_{i+2} + \ldots \right\}. \tag{13}$$

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* The complete integral of this equation when $n = -\,2i\,(2i+1)$, i being an integer, is

$$M_i = \mu^{2i} \left(\frac{d}{d\mu}\frac{1}{\mu} \right)^{2i} \mu^{2i}\,[K(1+\mu)^{2i} + K'\,(1-\mu)^{2i}],$$

where K and K' are the arbitrary constants.

The upper sign is to be used when i is even, the lower when odd. From (10) we obtain

$$M_1^2 = \tfrac{4}{5} - \tfrac{4}{7} M_1 + \tfrac{27}{35} M_2. \tag{14}$$

From this and the equation $r^2 f(\Sigma . V_i M_i) = \Sigma . T_i M_i$, pursuing the approximation to quantities of the second order, we get these expressions for T_i,

$$\left.\begin{aligned} r^{-2} T_0 &= f(V_0) + \tfrac{2}{5} V_1^2 f''(V_0) + \ldots, \\ r^{-2} T_1 &= V_1 f'(V_0) - \tfrac{2}{7} V_1^2 f''(V_0) + \ldots, \\ r^{-2} T_2 &= V_2 f'(V_0) + \tfrac{27}{70} V_1^2 f''(V_0) + \ldots \end{aligned}\right\} \tag{15}$$

If quantities of the second order are neglected, the two differential equations to be integrated are

$$\left.\begin{aligned} &\frac{d . r^2 \dfrac{dV_0}{dr}}{dr} + r^2 f(V_0) = 0, \\ &\frac{d . r^2 \dfrac{dV_1}{dr}}{dr} - 6 V_1 + r^2 V_1 f'(V_0) = 0. \end{aligned}\right\} \tag{16}$$

To pursue the analysis farther would require a knowledge of the form $f(V)$.

13. However, when the point r, μ, ω is without the surface of the earth, (11) can be integrated. Supposing the point not to partake in the motion of rotation, the centrifugal force must be neglected, and, since $\rho = 0$, generally $T_i = 0$; consequently (11) becomes

$$\frac{d . r^2 \dfrac{d\breve{V}_i}{dr}}{dr} - 2i(2i+1) \breve{V}_i = 0. \tag{17}$$

The integral of this is

$$\breve{V}_i = a_i r^{2i} + b_i r^{-(2i+1)},$$

in which $a_i = 0$; otherwise the earth's attraction would be infinitely great at an infinite distance. From the equation $V = \Sigma . V_i M_i$, by giving i successively the values 0, 1, 2, etc., we have, then,

$$\breve{V} = b_0 r^{-1} + b_1 r^{-3}(1 - 3\mu^2) + b_2 r^{-5}(1 - 10\mu^2 + \tfrac{35}{3}\mu^4) + \ldots \tag{18}$$

We have written V in this case $\breve{V}$. If r, μ, ω is situated in the atmosphere or at the surface, and revolves with the earth, we have only to add to $\breve{V}$ the potential of the centrifugal force $\dfrac{2\pi^2}{T^2} r^2 (1 - \mu^2)$, and may then call it $\bar{V}$; and for points within, write $\hat{V}$.

14. The integration of (11) introduces an indefinite number of constants, of which the superfluous are to be eliminated in the usual way of treating partial differential equations, viz., by equations of limitation; which, in the present question, are the equations having place at the limit of the attracting mass. At this limit $\frac{\partial V}{\partial r}, \frac{\partial V}{\partial \mu}, \frac{\partial V}{\partial \omega}$, derived from the integration of (5), must have the same value, whether ρ in that equation equals zero or its value at the surface. Consequently we have

$$\frac{\partial(\hat{V}-\overline{V})}{\partial r}=0, \quad \frac{\partial(\hat{V}-\overline{V})}{\partial \mu}=0, \quad \frac{\partial(\hat{V}-\overline{V})}{\partial \omega}=0. \tag{19}$$

Hydrostatics furnishes two other equations,

$$\hat{V}+B=0, \quad \overline{V}+B'=0. \tag{20}$$

These five are all equations to the same surface, of which it must be noticed that they are not independent, but any one is a consequence of the other four. They will give all the conditions needed to eliminate the superfluous arbitrary constants. The last of (19) is, since V does not involve ω, an identical equation; select then the first of (19) and the two of (20) for the purpose of elimination. It is evident from $V=\Sigma . V_i M_i$ that each one of these equations to the surface can be transformed to this shape: $r=\Sigma . c_i M_i$, c_i being a constant and of the i^{th} order with respect to the centrifugal force. By substituting this value of r in equations (19) and (20) and equating the coefficients of M_i generally to zero, and rejecting the two involving B and B', we have, pursuing the approximation to the i^{th} order, $3i+1$ equations. The constants they involve are the constants resulting from the integration of (11), $2i+2$ in number, and the constants c_i, $i+1$ in number, in all $3i+3$. Thus two constants are left indeterminate, which is as it should be, since the units of length and density are arbitrary. We may assume, then, $c_0=1$;* making this substitution in $\hat{V}+B=0$, using accents to denote differentiation and the putting r afterwards equal to unity, and taking account of quantities of the second order, we have

$$\left.\begin{aligned} V_0' c_1 + V_1 - \tfrac{2}{7} c_1 (V_0'' c_1 + 2 V_1') &= 0, \\ V_0' c_2 + V_2 + \tfrac{27}{70} c_1 (V_0'' c_1 + 2 V_1') &= 0. \end{aligned}\right\} \tag{21}$$

Treating $\overline{V}+B'=0$ in the same way, two similar equations result, which may be derived from (21) by making

$$V_0=b_0 r^{-1}+\frac{4\pi^2}{3T^2}r^2, \quad V_1=b_1 r^{-3}+\frac{2\pi^2}{3T^2}r^2, \quad \text{and } V_2=b_2 r^{-5}.$$

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* An injudicious proceeding, as the homogeneity of the formulas is thereby lost.

Thus

$$\left.\begin{aligned}\left(-b_0+\frac{8\pi^2}{3T^2}\right)c_1+b_1+\frac{2\pi^2}{3T^2}-\tfrac{2}{7}c_1\left(2b_0c_1-6b_1+\frac{8\pi^2}{3T^2}\right)&=0,\\ -b_0c_2+b_2+\tfrac{27}{70}c_1\left(2b_0c_1-6b_1+\frac{8\pi^2}{3T^2}\right)&=0.\end{aligned}\right\}\qquad(22)$$

From the equation $\dfrac{\partial(\hat{V}-\bar{V})}{\partial r}=0$, we also have

$$\left.\begin{aligned}V_0'+\tfrac{2}{5}c_1(V_0'''c_1+2V_1'')&=-b_0+\frac{8\pi^2}{3T^2}+\tfrac{2}{5}c_1\left(-6b_0c_1+24b_1+\frac{8\pi^2}{3T^2}\right),\\ V_0''c_1+V_1'-\tfrac{2}{7}c_1(V_0'''c_1+2V_1'')&=\left(2b_0+\frac{8\pi^2}{3T^2}\right)c_1-3b_1+\frac{4\pi^2}{3T^2}\\ &\quad-\tfrac{2}{7}c_1\left(-6b_0c_1+24b_1+\frac{8\pi^2}{3T^2}\right),\\ V_0''c_2+V_2'+\tfrac{27}{70}c_1(V_0'''c_1+2V_1'')&=2b_0c_2-5b_2+\tfrac{27}{70}c_1\left(-6b_0c+24b_1+\frac{8\pi^2}{3T^2}\right).\end{aligned}\right\}\qquad(23)$$

If we make

$$\frac{4\pi^2}{T^2}=q\,[b_0(1-3c_1)+3b_1],$$

equations (22) and (23) can be reduced to the following simpler forms:

$$\left.\begin{aligned}b_0&=-(1+\tfrac{2}{3}q+\tfrac{1}{9}q^2-\tfrac{4}{3}qc_1+\tfrac{36}{5}c_1^2)[V_0'+\tfrac{2}{5}c_1(V_0'''c_1+2V_1'')],\\ b_1&=[(1-\tfrac{4}{21}q)c_1-\tfrac{8}{7}c_1^2-\tfrac{1}{6}q+\tfrac{1}{12}q^2]\,b_0,\\ b_2&=(c_2-\tfrac{9}{14}qc_1+\tfrac{54}{35}c_1)\,b_0,\end{aligned}\right\}\qquad(24)$$

$$\left.\begin{aligned}V_0''c_1+V_1'-\tfrac{2}{7}c_1(V_0'''c_1+2V_1'')-[(1-\tfrac{32}{21}q)c_1+\tfrac{12}{7}c_1^2-\tfrac{5}{6}q-\tfrac{5}{36}q^2]\,V_0'&=0,\\ V_2''c_2+V_2'+\tfrac{27}{70}c_1(V_0'''c_1+2V_1'')-(3c_2-\tfrac{27}{14}qc_1+\tfrac{27}{35}c_1^2)\,V_0'&=0.\end{aligned}\right\}\qquad(25)$$

15. In this article we shall neglect quantities of the order of q^2. Let $\frac{1}{3}w$ denote the quantity obtained by dividing the second member of the second equation of (23) by the second member of the first equation; then

$$w=3\,\frac{-3b_1+\dfrac{4\pi^2}{3T^2}+2b_0c_1}{b_0}=6c_1+q-9\,\frac{b_1}{b_0};$$

from the second equation of (24), $3c_1=3\,\dfrac{b_1}{b_0}+\frac{1}{2}q=e$, the approximate value of the earth's compression, as is clear from the equation

$$r=1+c_1(1-3\mu^2)+\ldots.$$

Hence, by addition,

$$e+w=6c_1+\tfrac{3}{2}q-6\,\frac{b_1}{b_0}=\tfrac{5}{2}q.\qquad(26)$$

The relation enunciated by this equation is known as Clairaut's Theorem.

16. The equation of the earth's surface is

$$\left.\begin{aligned} r &= 1 + c_1(1-3\mu^2) + c_2(1-10\mu^2 + \tfrac{35}{3}\mu^4) + \ldots \\ &= (1 + c_1 + c_2)\{1 - [3c_1(1-c_1) + 10c_2]\mu^2 + \tfrac{35}{3}c_2\mu^4 + \ldots\} \\ &= a[1 + e_1 \sin^2\theta + e_2 \sin^4\theta + \ldots]; \end{aligned}\right\} \tag{27}$$

in which θ is the geocentric latitude. If e represents the compression, or part of its own length by which the equatorial radius exceeds the polar,

$$e = -e_1 - e_2 = 3c_1(1-c_1) - \tfrac{5}{3}c_2. \tag{28}$$

17. Denoting the normal or astronomical latitude by θ', we have

$$\theta' = \theta - \tan^{-1}\frac{dr}{rd\theta} = \theta - (e_1 + e_2 - \tfrac{1}{2}e_1^2)\sin 2\theta + (\tfrac{1}{2}e_2 - \tfrac{1}{4}e_1^2)\sin 4\theta, \tag{29}$$

the inversion of which is

$$\theta = \theta' + (e_1 + e_2 - \tfrac{1}{2}e_1^2)\sin 2\theta' + (\tfrac{1}{2}e_2 + \tfrac{3}{4}e_1^2)\sin 4\theta'. \tag{30}$$

18. A line geodetically measured on the earth's surface is clearly the shortest possible; hence, if ds is the element of the curve, by the principles of the Calculus of Variations

$$\delta\int ds + \lambda\delta V = 0,$$

λ being the indeterminate multiplier of δV. Also,

$$ds^2 = dr^2 + \frac{r^2 d\mu^2}{1-\mu^2} + r^2(1-\mu^2)\,d\omega^2.$$

The coefficients of δr, $\delta\mu$, $\delta\omega$ each equal zero; retaining only that of $\delta\omega$, as sufficient for our purpose, we have

$$d\left[r^2(1-\mu^2)\frac{d\omega}{ds}\right] = 0, \quad \text{or} \quad r^2(1-\mu^2)\frac{d\omega}{ds} = ah.$$

That is, the sine of the angle made by the curve with the meridians varies inversely as the distance from the earth's axis. Hence,

$$\frac{d\omega}{d\mu} = \frac{ah\sqrt{1 + \left(\frac{dr}{rd\mu}\right)^2(1-\mu^2)}}{(1-\mu^2)\sqrt{r^2(1-\mu^2) - a^2h^2}} \quad \text{and} \quad \frac{ds}{d\mu} = \frac{r^2\sqrt{1 + \left(\frac{dr}{rd\mu}\right)^2(1-\mu^2)}}{\sqrt{r^2(1-\mu^2) - a^2h^2}}.$$

Taking account only of terms of the first order with respect to q,

$$d\omega = \frac{h\,d\mu}{(1-\mu^2)\sqrt{1-h^2-\mu^2}} - \frac{he_1\mu^2 d\mu}{(1-h^2-\mu^2)^{\frac{3}{2}}};$$

which, integrated, gives

$$\omega = C - \tfrac{1}{2}\sin^{-1}\frac{1-\frac{1+h^2}{1-h^2}\mu^2}{1-\mu^2} + he_1\left[\sin^{-1}\frac{\mu}{\sqrt{1-h^2}} - \frac{\mu}{\sqrt{1-h^2-\mu^2}}\right]. \qquad (31)$$

This, then, is the equation to the curve; if $\omega_{,}$, $\omega_{,,}$ are the extreme values of ω, and $\mu_{,}$, $\mu_{,,}$ those of μ, h can be found from the expression for $\omega_{,,}-\omega_{,}$. If ε is the angle made by the curve with the meridian at the commencement, then $h = \cos\theta_{,}\sin\varepsilon\,(1+e_1\sin^2\theta_{,})$, and, as affording an approximate value of ε, we have

$$\cot\varepsilon = \frac{\tan\theta_{,,}\cos\theta_{,}}{\sin(\omega_{,,}-\omega_{,})} - \sin\theta_{,}\cot(\omega_{,,}-\omega_{,}).$$

For the length of the curve,

$$ds = a\left[\frac{1+e_1\mu^2}{\sqrt{1-h^2-\mu^2}} - \frac{h^2e_1\mu^2}{(1-h^2-\mu^2)^{\frac{3}{2}}}\right]d\mu,$$

which, integrated, gives

$$s = C + a\left\{\left[1+(1+h^2)\frac{e_1}{2}\right]\sin^{-1}\frac{\mu}{\sqrt{1-h^2}} - \frac{e_1}{2}\frac{(1+h^2-\mu^2)\mu}{\sqrt{1-h^2-\mu^2}}\right\}. \qquad (32)$$

If $h=0$, this expression gives the length of any arc of the meridian, but in this case

$$ds = \sqrt{dr^2+r^2d\theta^2} = r\left(1+\frac{e_1^2}{2}\sin^2 2\theta\right)d\theta, = a\left[1+e_1\sin^2\theta + e_2\sin^4\theta + \frac{e_1^2}{2}\sin^2 2\theta\right)d\theta,$$

which, integrated, gives

$$s = C + a\left[\left(1+\frac{e_1}{2}+\frac{e_1^2}{4}+\tfrac{3}{8}e_2\right)\theta - \frac{e_1+e_2}{4}\sin 2\theta - \frac{2e_1^2-e_2}{32}\sin 4\theta\right]. \qquad (33)$$

19. All areas on the earth's surface, bounded by lines whose equations are (31), can be divided into a finite number of parts, each contained by an arc of a meridian, an arc of a parallel of latitude, and a line whose equation is (31). Let A denote the area of this, then ds being the element of the meridian, $A = \int\!\!\int r\cos\theta\,d\omega\,ds$, or, neglecting quantities of the second order, $A = \int\!\!\int r^2\,d\mu\,d\omega$. If this is integrated along a meridian, the result is

$$A = a^2\int[(1+\tfrac{2}{3}e_1\mu^2_{,,})\mu_{,,} - (1+\tfrac{2}{3}\mu^2)\mu]\,d\omega$$

$$= a^2(1+\tfrac{2}{3}e_1\mu^2_{,,})\mu_{,,}(\omega_{,,}-\omega_{,}) - a^2h\int\left[\frac{1+\frac{2}{3}e_1\mu^2}{(1-\mu^2)\sqrt{1-h^2-\mu^2}} - \frac{e_1\mu^2}{(1-h^2-\mu^2)^{\frac{3}{2}}}\right]\mu\,d\mu \qquad (34)$$

$$= a^2(1+\tfrac{2}{3}e_1\mu^2_{,,})\mu_{,,}(\omega_{,,}-\omega_{,}) + \frac{a^2h}{2}\left[\frac{1+\frac{2}{3}e_1}{h}\tan^{-1}\frac{\sqrt{1-h^2-\mu}}{h} + \tfrac{2}{3}e_1\frac{4(1-h^2)-\mu^2}{\sqrt{1-h^2-\mu^2}} + C\right].$$

The arbitrary constant C in all these formulas is determined by the condition that the length or area must vanish when the beginning and end of the geodetic line coincide.*

20. Let F denote the force of gravity at the surface, then

$$F = \frac{\partial V}{\partial r}\sqrt{1 + \left(\frac{dr}{rd\theta}\right)^2} = \frac{\partial V}{\partial r}[1 + 18\, c_1^2 \mu^2 (1-\mu^2)].$$

If A_0, A_1, A_2 are the numerical values of the members of equations (23), then

$$\begin{aligned} F &= (A_0 + A_1 + A_2)\left[1 - \frac{3A_1 + 10A_2 - 18\, V_0^1 c_1^2}{A_0 + A_1}\mu^2 + \frac{\frac{35}{3} A_2 - 18\, V_0^1 c_1^2}{A_0}\mu^4\right], \\ &= F_0[1 + w_1 \mu^2 + w_2 \mu^4 + \ldots]. \end{aligned}$$

21. Thus far the general theory of the subject. We shall now assume some particular law of density. Suppose that the matter of which the earth is composed is compressible inversely as its density. This gives $d\rho = \frac{m^2}{4\pi}\frac{dp}{\rho}$, m being a constant. Substituting for dp its value ρdV, and integrating, $\rho = \frac{m^2}{4\pi} V$. No constant is added, because it may be supposed contained in V. Then

$$f(V) = m^2 V - \frac{8\pi^2}{T^2}$$

and from (15) generally

$$T_i = m^2 r^3 V_i;$$

but

$$T_0 = m^2 r^3 \left(V_0 - \frac{8\pi^2}{m^2 T^2}\right),$$

and thus (11) becomes

$$\frac{d \,.\, r^2 \frac{dV_i}{dr}}{dr} - [2i(2i+1) - m^2 r^2]\, V_i = 0\,. \tag{36}$$

In integrating, the part of V_i involving negative powers of r may be neglected, since it belongs to $\breve{V}$. If a_i is an arbitrary constant, and f^{2i} denotes the operation $\frac{d}{dr}\,\frac{1}{r}$ performed $2i$ times, the integral of (36) is

$$V_i = a_i r^{2i-1} f^{2i} (\sin mr)\,.$$

* Equation (34) in the original memoir is erroneous; the correct form is given here.

It may also be obtained thus: put (36) under this form

$$\frac{d^2 rV_i}{dr^2}+\left[m^2-\frac{2i(2i+1)}{r^2}\right]rV_i=0. \tag{37}$$

Assume

$$rV_i=P\sin mr+P'\cos mr;$$

which, by substitution, gives

$$\frac{d^2(P\pm P')}{dr^2}\pm 2m\frac{d(P\mp P')}{dr}-\frac{2i(2i+1)}{r^2}(P\pm P')=0.$$

Make

$$P\pm P'=\beta_0\pm\beta_1 r^{-1}+\beta_2 r^{-2}\pm\beta_3 r^{-3}+\ldots;$$

then this equation results

$$\pm 2m(n+1)\beta_{n+1}+[n(n+1)-2i(2i+1)]\beta_n=0,$$

whence

$$\beta_{n+1}=\mp\frac{(n-2i)(n+2i+1)}{2(n+1)m}\beta_n,$$

the upper sign being taken when n is even, the lower when it is odd. Then making $\beta_0=\pm m^{2i}a_i$, in order to agree with the expression $V_i=a_i r^{2i-1}f^{2i}(\sin mr)$, we have

$$\beta_n=\pm\frac{(2i-n+1)\ldots(2i+n)}{1.2\ldots n.2^n}m^{2i-n}a_i; \tag{38}$$

the upper sign having place when $2i-n$ is of the forms $4\nu+2$, $4\nu+3$, the lower when it is of the forms 4ν, $4\nu+1$.

$$V=\frac{8\pi^2}{m^2T^2}+a_0r^{-1}f^0(\sin mr)+a_1rf^2(\sin mr)(1-3\mu^2)$$
$$+a_2r^3f^4(\sin mr)(1-10\mu^2+\tfrac{35}{3}\mu^4)+\ldots, \tag{39}$$

or, expanding $f^{2i}(\sin mr)$ by using (38),

$$V=\frac{8\pi^2}{m^2T^2}+a_0\frac{\sin mr}{r}+a_1\left[\left(\frac{3}{r^2}-m^2\right)\frac{\sin mr}{r}-\frac{3m}{r}\frac{\cos mr}{r}\right](1-3\mu^2) \tag{40}$$
$$+a_2\left[\left(\frac{105}{r^4}-\frac{45m^2}{r^2}+m^4\right)\frac{\sin mr}{r}-\left(\frac{105m}{r^3}-\frac{10m^3}{r}\right)\frac{\cos mr}{r}\right](1-10\mu^2+\tfrac{35}{3}\mu^4)+\ldots$$

22. Since V contains a constant, the term $\frac{8\pi^2}{m^2T^2}$ may be neglected except in finding the value of the density. Moreover, for simplicity, let $a_0=1$ and $\frac{V_0'}{m^2V_0}=-H$; then, from (36) and (40), we derive

$$\left.\begin{aligned}
V_0''&=(2H-1)\,m^2V_0,\\
V_0'''&=[2+(m^2-6)H]\,m^2V_0,\\
V_1&=a_1(3H-1)\,m^2V_0,\\
V_1'&=a_1[3+(m^2-9)H]\,m^2V_0,\\
V_1''&=a_1[m^2-12-(5m^2-36)H]\,m^2V_0,\\
V_2&=a_2[m^2-35-(10m^2-105)H]\,m^2V_0,\\
V_2'&=a_2[-10m^2+175-(m^4-65m^2+525)H]\,m^2V_0.
\end{aligned}\right\} \tag{41}$$

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By substituting these expressions in (21) and (25), neglecting quantities of the second order, they become, after removing the factor $m^2 V_0$,

$$\begin{aligned}(3H-1)\,a_1 - H c_1 &= 0\,,\\ [3+(m^2-9)H]\,a_1 + (3H-1)\,c_1 &= \tfrac{5}{6} qH\,.\end{aligned}$$

Whence,

$$a_1 = \tfrac{5}{6} q \frac{H^2}{m^2H^2-(3H-1)}\,, \quad c_1 = \tfrac{5}{6} q \frac{H(3H-1)}{m^2H^2-(3H-1)}\,. \tag{42}$$

23. Represent the volume of the earth by v, its superficial density by R, its mean density by R'. Then, neglecting quantities of the second order, $v = \frac{4\pi}{3}$; and, if (V) denote the value of V at the surface, $R = \frac{m^2}{4\pi}(V)$. Since b_0 is the mass of the earth, $R' = \frac{b_0}{v} = \frac{3b_0}{4\pi}$. Hence $\frac{m^2(V)}{b_0} = \frac{3R}{R'}$, or, putting for b_0 its value from (24),

$$\frac{V_0'}{m^2(V)} = -\frac{R'}{3R(1+\frac{2}{3}q)}\,.$$

If, in the expression for V, we make $3\mu^2 = 1$, and, consequently, $r = 1$;

$$(V) = \frac{8\pi^2}{m^2T^2} + V_0 = V_0 - \frac{2q}{m^2} V_0'\,,$$

then

$$\frac{H}{1+2qH} = \frac{R'}{3R(1+\frac{2}{3}q)}\,,$$

or

$$H = \frac{R'}{3R - 2q(R'-R)}\,. \tag{43}$$

To find m we have, since $H = \frac{-V_0'}{m^2 V_0}$,

$$\frac{1}{m}\left(\frac{1}{m} - \cot m\right) = H\,. \tag{44}$$

24. In order to test the preceding theory by numerical calculation, we adopt the following values for q, R, R', the best we can find:

$$q = \tfrac{1}{288}\,, \quad R = 2.55\,, \quad R' = 5.67\,.$$

We shall mark with an accent the numbers of the formulas from which the numerical values of the following quantities are obtained :*

$$
\begin{aligned}
&(43)' \quad H = 0.7432817\,, \\
&(44)' \quad \begin{cases} m = 146^\circ\, 27'\, 56''.2\,, \\ = 2.556307\,, \end{cases} \\
&(42)' \quad \begin{cases} a_1 = 0.0006715666\,, \\ c_1 = 0.0011111845\,, \end{cases} \\
&(41)' \quad V_0'' = 0.48656 m^2 V_0\,,
\end{aligned}
\qquad
(41)' \begin{cases}
V_0''' = 2.39744 m^2 V_0\,, \\
V_1 = 1.2298451 m^2 V_0 a_1\,, \\
V_1' = 1.1675917 m^2 V_0 a_1\,, \\
V_1'' = -2.99279 m^2 V_0 a_1\,, \\
V_2 = 1.00801 m^2 V_0 a_2\,, \\
V_2' = 3.40342 m^2 V_0 a_2\,.
\end{cases}
$$

To obtain the values of a_1, a_2, c_1, c_2 true to quantities of the order of q^2, by substituting the preceding in (21) and (25), we have

$$
\begin{aligned}
&(21)' \quad 0.7432817 c_1 - 1.2298451 a_1 + 0.0000006695 = 0\,, \\
&(25)' \quad 1.2298451 c_1 + 1.1675917 a_1 - 0.0021543100 = 0\,, \\
&(21)' \quad 0.74328 c_2 \quad - 1.00801 a_2 \quad - 0.0000009039 = 0\,, \\
&(25)' \quad 2.71641 c_2 \quad + 3.40342 a_2 \quad - 0.0000054038 = 0\,.
\end{aligned}
$$

The solution gives

$$
\begin{aligned}
a_1 &= 0.0006730400\,, & c_1 &= 0.0011127213\,, \\
a_2 &= 0.0000002964\,, & c_2 &= 0.0000016180\,.
\end{aligned}
$$

$$
\begin{aligned}
&(24)' \quad \check{V} = b_0 [r^{-1} + 0.0005328715 r^{-3} (1 - 3\mu^2) + 0.0000010445 r^{-5} (1 - 10\mu^2 + \tfrac{35}{3}\mu^4)]\,, \\
&(27)' \quad r = a[1 - 0.003350630 \sin^2\theta + 0.000018877 \sin^4\theta]\,, \\
&(28)' \quad e = 0.003331753 = \frac{1}{300.1423}\,, \\
&(29)' \quad \theta' = \theta + 688.''3811 \sin 2\theta + 1.''3679 \sin 4\theta\,, \\
&(30)' \quad \theta = \theta' - 688.''3811 \sin 2\theta' + 3.''6836 \sin 4\theta'\,, \\
&(33)' \quad s = a[0.998334571\, \theta + 0.000832938 \sin 2\theta - 0.000000112 \sin 4\theta]\,, \\
&(35)' \quad F = F_0 [1 + 0.005406990 \sin^2\theta - 0.000041419 \sin^4\theta]\,.
\end{aligned}
$$

The following table contains the values of ρ and of e, the compression of the surfaces of level, for every tenth of the equatorial radius, calculated from the equations

$$
\rho = \frac{m^2}{4\pi} V = \frac{R \sin mr}{r \sin m}\,, \quad \text{and} \quad e = -\frac{3 V_1}{V_0'} = 3a_1 \left[\frac{3}{r} - \frac{m}{\frac{1}{mr} - \cot mr}\right].
$$

$\frac{r}{a}$	ρ	e	$\frac{r}{a}$	ρ	e
0.0	11.800	0	0.6	7.688	1 – 587th
0.1	11.672	1 – 3773th	0.7	6.437	1 – 489th
0.2	11.292	1 – 1880th	0.8	5.133	1 – 413th
0.3	10.677	1 – 1242th	0.9	3.822	1 – 351th
0.4	9.848	1 – 919th	1.0	2.550	1 – 300th
0.5	8.839	1 – 722th			

* The numbers following this in the original memoir are erroneous; they are here rectified.

MEMOIR No. 4.

Ephemeris of the Great Comet of 1858.

(Astronomische Nachrichten, Vol. 64, pp. 181-190, 1865.)

The coordinates given in the following ephemeris are unaffected with aberration; the constant intended to be used is that of Struve. The columns $\Delta\alpha$, $\Delta\delta$, contain the excess of the present ephemeris over that used for comparison.*

Wash. 0h		True α	Δ α	True δ	Δ δ	Log r	Log Δ
		° ′ ″	″	° ′ ″	″		
1858, June	6	141 14 9.88	—2.66	+24 13 51.41	—8.47	0.33754	0.39511
	9	141 15 38.45	2.56	24 33 33.59	7.65	0.32900	0.39669
	12	141 20 1.24	2.51	24 52 32.08	6.95	0.32024	0.39795
	15	141 27 11.39	2.52	25 10 52.97	6.22	0.31124	0.39887
	18	141 37 2.40	2.45	25 28 42.01	5.34	0.30198	0.39942
	21	141 49 28.00	2.30	25 46 4.62	4.60	0.29247	0.39959
	24	142 4 22.65	2.39	26 3 5.98	3.82	0.28267	0.39936
	27	142 21 41.87	2.34	26 19 51.10	3.29	0.27259	0.39871
	30	142 41 22.12	2.31	26 36 25.07	2.48	0.26219	0.39762
July	3	143 3 20.57	2.29	26 52 52.10	1.93	0.25147	0.39609
	6	143 27 35.01	2.21	27 9 17.53	1.32	0.24041	0.39407
	9	143 54 3.55	2.15	27 25 46.19	0.72	0.22897	0.39155
	12	144 22 44.48	2.22	27 42 23.55	—0.17	0.21716	0.38851
	15	144 53 36.34	2.33	27 59 15.31	+0.48	0.20493	0.38492
	18	145 26 38.65	2.45	28 16 27.20	1.23	0.19226	0.38074
	21	146 1 52.22	2.55	28 34 4.66	1.90	0.17914	0.37595
	24	146 39 19.33	2.65	28 52 13.27	2.52	0.16552	0.37052
	27	147 19 4.27	2.87	29 10 58.54	3.07	0.15138	0.36440
	30	148 1 13.43	3.12	29 30 26.24	3.69	0.13669	0.35756
Aug.	2	148 45 55.13	3.29	29 50 42.07	4.40	0.12142	0.34994
	5	149 33 19.76	3.58	30 11 52.17	5.08	0.10552	0.34149
	8	150 23 40.13	4.02	30 34 2.98	5.75	0.08896	0.33214
	11	151 17 12.15	4.45	30 57 21.02	6.35	0.07170	0.32181
	14	152 14 15.79	5.03	31 21 52.34	6.97	0.05372	0.31041
	17	153 15 17.36	5.59	31 47 41.92	7.72	0.03498	0.29785
	20	154 20 51.06	6.08	32 14 52.80	8.41	0.01545	0.28402
	23	155 31 41.08	6.75	32 43 25.42	9.06	9.99514	0.26878
	26	156 48 45.37	7.59	33 13 15.90	9.74	9.97404	0.25196
	29	158 13 19.50	8.16	33 44 14.36	10.59	9.95219	0.23337
Sept.	1	159 47 1.10	9.15	34 16 0.63	11.49	9.92968	0.21279
	2	160 20 37.81	9.30	34 26 41.02	11.50	9.92205	0.20544
	3	160 55 35.02	—9.55	+34 37 21.35	+11.61	9.91437	0.19783

* It should be stated that this ephemeris is constructed from the final theory of Memoir No. 6, pp. 25-58.

Wash. 0h		True α	Δα	True δ	Δδ	Log r	Log Δ
		° ′ ″	″	° ′ ″	″		
1858, Sept.	4	161 31 58.87	— 9.83	+34 47 59.31	+11.79	9.90665	0.18994
	5	162 9 56.20	10.14	34 58 32.25	11.90	9.89889	0.18177
	6	162 49 34.55	10.44	35 8 57.37	12.12	9.89112	0.17330
	7	163 31 2.11	10.75	35 19 11.02	12.34	9.88334	0.16451
	8	164 14 27.99	11.01	35 29 8.94	12.57	9.87557	0.15540
	9	165 0 2.13	11.25	35 38 46.22	12.76	9.86782	0.14595
	10	165 47 55.45	11.53	35 47 57.10	12.95	9.86011	0.13614
	11	166 38 19.94	11.79	35 56 34.79	13.14	9.85247	0.12595
	12	167 31 28.74	12.04	36 4 31.28	13.38	9.84492	0.11537
	13	168 27 36.18	12.33	36 11 37.43	13.60	9.83748	0.10439
	14	169 26 57.89	12.55	36 17 42.34	13.64	9.83019	0.09299
	15	170 29 50.86	12.82	36 22 33.31	13.72	9.82307	0.08114
	16	171 36 33.54	13.08	36 25 55.69	13.86	9.81616	0.06884
	17	172 47 25.71	13.29	36 27 32.35	13.98	9.80949	0.05607
	18	174 2 48.55	13.40	36 27 3.32	14.15	9.80310	0.04281
	19	175 23 4.32	13.56	36 24 5.57	14.25	9.79703	0.02906
	20	176 48 36.37	13.68	36 18 12.45	14.34	9.79133	0.01480
	21	178 19 48.74	13.74	36 8 53.06	14.42	9.78602	0.00003
	22	179 57 5.84	13.83	35 55 32.06	14.56	9.78115	9.98476
	23	181 40 51.78	13.72	35 37 28.78	14.63	9.77677	9.96898
	24	183 31 29.48	13.64	35 13 57.05	14.74	9.77291	9.95272
	25	185 29 19.90	13.49	34 44 4.69	14.74	9.76960	9.93601
	26	187 34 40.58	13.27	34 6 53.07	14.74	9.76687	9.91889
	27	189 47 44.44	12.94	33 21 17.77	14.79	9.76477	9.90143
	28	192 8 38.03	12.56	32 26 8.32	14.73	9.76329	9.88372
	29	194 37 19.97	12.03	31 20 10.26	14.76	9.76247	9.86587
	30	197 13 39.26	11.52	30 2 6.42	14.61	9.76230	9.84804
Oct.	1	199 57 13.94	10.91	28 30 41.24	14.45	9.76280	9.83042
	2	202 47 29.98	10.13	26 44 44.87	14.19	9.76395	9.81324
	3	205 43 40.69	9.41	24 43 20.34	13.68	9.76574	9.79678
	4	208 44 47.46	8.52	22 25 51.80	13.22	9.76816	9.78137
	5	211 49 40.83	7.58	19 52 13.03	12.47	9.77118	9.76736
	6	214 57 2.54	6.40	17 2 57.06	11.59	9.77477	9.75514
	7	218 5 29.17	5.79	13 59 22.54	10.65	9.77890	9.74507
	8	221 13 35.77	4.75	10 43 36.87	9.56	9.78352	9.73748
	9	224 19 58.83	3.63	7 18 32.39	8.37	9.78862	9.73264
	10	227 23 20.16	2.69	3 47 36.33	7.16	9.79413	9.73070
	11	230 22 30.47	2.01	+0 14 34.17	5.45	9.80002	9.73171
	12	233 16 30.95	1.28	—3 16 49.39	4.80	9.80625	9.73558
	13	236 4 34.32	0.74	6 43 9.47	3.80	9.81279	9.74211
	14	238 46 5.66	—0.31	10 1 35.21	2.94	9.81958	9.75101
	15	241 20 41.21	+0.09	13 9 57.92	2.17	9.82660	9.76194
	16	243 48 7.80	0.39	16 6 52.25	1.76	9.83381	9.77453
	17	246 8 21.80	0.55	18 51 33.65	1.24	9.84118	9.78843
	18	248 21 27.23	0.53	21 23 50.16	0.96	9.84868	9.80331
	19	250 27 33.93	0.62	23 43 55.37	0.90	9.85628	9.81886
	20	252 26 56.04	+0.46	25 52 21.11	0.85	9.86395	9.83485
	21	254 19 51.19	—0.01	27 49 49.80	1.00	9.87168	9.85107
	22	256 6 39.54	0.40	29 37 10.18	1.12	9.87944	9.86734
	23	257 47 42.34	0.64	31 15 13.14	1.22	9.88722	9.88355
	24	259 23 20.88	1.15	32 44 48.82	1.35	9.89500	9.89958
	25	260 53 56.93	—1.58	—34 6 45.50	+1.54	9.90277	9.91538

Wash. 0h		True α	Δ α	True δ	Δ δ	Log r	Log Δ
		° ′ ″	″	° ′ ″	″		
1858, Oct.	26	262 19 51.45	—1.98	—35 21 48.24	+1.68	9.91051	9.93087
	27	263 41 24.71	2.57	36 30 38.26	1.87	9.91821	9.94602
	28	264 58 56.27	3.21	37 33 53.30	2.15	9.92586	9.96081
	29	266 12 44.50	3.86	38 32 7.34	2.35	9.93346	9.97521
	30	267 23 6.42	4.47	39 25 50.85	2.44	9.94100	9.98923
	31	268 30 18.13	5.16	40 15 30.68	2.68	9.94847	0.00285
Nov.	1	269 34 34.73	5.84	41 1 31.33	2.85	9.95587	0.01608
	4	272 32 6.07	8.02	43 1 0.91	3.36	9.97760	0.05349
	7	275 10 36.95	10.27	44 38 22.12	3.71	9.99857	0.08767
	10	277 34 14.07	12.42	45 59 3.78	3.80	0.01876	0.11892
	13	279 46 8.19	14.54	47 7 0.96	4.17	0.03815	0.14754
	16	281 48 46.92	16.56	48 5 4.46	4.12	0.05676	0.17382
	19	283 44 5.42	18.49	48 55 19.62	4.03	0.07462	0.19801
	22	285 33 34.41	20.64	49 39 20.44	3.94	0.09176	0.22034
	25	287 18 26.11	22.73	50 18 18.84	3.82	0.10821	0.24100
	28	288 59 38.31	24.46	50 53 10.72	3.58	0.12400	0.26015
Dec.	1	290 37 56.18	26.31	51 24 39.95	3.41	0.13918	0.27794
	4	292 13 55.21	27.98	51 53 22.34	3.24	0.15377	0.29450
	7	293 48 3.40	29.38	52 19 46.72	2.98	0.16782	0.30992
	10	295 20 42.09	31.00	52 44 16.96	2.51	0.18135	0.32431
	13	296 52 8.76	32.41	53 7 12.21	2.35	0.19440	0.33775
	16	298 22 37.88	33.86	53 28 48.88	1.92	0.20699	0.35031
	19	299 52 22.11	35.15	53 49 20.28	1.67	0.21915	0.36205
	22	301 21 32.66	36.28	54 8 57.88	1.09	0.23090	0.37305
	25	302 50 19.37	37.35	54 27 51.54	0.68	0.24227	0.38334
	28	304 18 50.80	38.34	54 46 10.16	+0.27	0.25328	0.39298
	31	305 47 13.68	39.38	55 4 1.96	—0.23	0.26394	0.40200
1859, Jan.	3	307 15 33.18	40.37	55 21 34.49	0.78	0.27429	0.41045
	6	308 43 53.24	41.23	55 38 54.59	1.51	0.28432	0.41835
	9	310 12 17.04	41.91	55 56 8.15	2.13	0.29407	0.42573
	12	311 40 47.05	42.66	56 13 20.59	2.77	0.30354	0.43264
	15	313 9 26.42	43.33	56 30 36.49	3.42	0.31275	0.43908
	18	314 38 18.98	43.70	56 48 0.05	4.09	0.32171	0.44510
	21	316 7 28.53	44.10	57 5 35.13	4.78	0.33044	0.45071
	24	317 36 59.54	44.34	57 23 25.61	5.56	0.33894	0.45594
	27	319 6 55.45	44.53	57 41 35.30	6.36	0.34722	0.46079
	30	320 37 19.46	44.73	58 0 8.20	7.16	0.35530	0.46530
Feb.	2	322 8 14.64	44.72	58 19 8.24	8.00	0.36319	0.46949
	5	323 39 43.18	44.61	58 38 39.00	8.87	0.37088	0.47335
	8	325 11 47.56	44.48	58 58 43.52	9.79	0.37840	0.47692
	11	326 44 30.89	44.19	59 19 24.71	10.72	0.38575	0.48022
	14	328 17 56.99	43.80	59 40 44.85	11.74	0.39294	0.48325
	17	329 52 10.91	43.29	60 2 46.12	12.77	0.39996	0.48604
	20	331 27 18.49	42.63	60 25 30.70	13.82	0.40684	0.48860
	23	333 3 25.63	41.78	60 49 0.77	14.89	0.41357	0.49094
	26	334 40 38.55	40.75	61 13 18.80	15.92	0.42017	0.49308
Mar.	1	336 19 3.23	39.57	61 38 27.12	17.08	0.42662	0.49503
	4	337 58 45.67	38.18	62 4 27.82	18.24	0.43296	0.49681
	7	339 39 52.15	—36.55	—62 31 22.53	—19.44	0.43916	0.49843

MEMOIR No. 5.

On the Reduction of the Rectangular Coordinates of the Sun Referred to the True Equator and Equinox of Date to those Referred to the Mean Equator and Equinox of the Beginning of the Year.

(Astronomische Nachrichten, Vol. 67, pp. 141-142, 1866.)

In computing an ephemeris of any planetary body, it is quite the easiest plan to get the heliocentric rectangular coordinates referred to fixed planes, as those defined by the mean equator and equinox of the beginning of Bessel's fictitious year, either of the current year or of the nearest tenth year. Then, by the addition of the sun's coordinates referred to the same planes, to obtain the geocentric rectangular coordinates, and from thence to proceed to the corresponding polar coordinates, which may be very readily changed to the true equator and equinox of date by using the three star constants f, g and G.

But the coordinates of the sun hitherto published in the various ephemerides have not been rigorously reduced to these planes.

The following method of reduction is offered as being quite simple, since it involves only the star constants in addition to the coordinates themselves.

Let R denote the sun's radius vector and α, δ its true right ascension and declination referred to the mean planes of the beginning of the year, and α', δ' the same referred to the true planes of date, and let X, Y, Z, X', Y', Z' be the corresponding rectangular coordinates.

Whence result these relations

$$\begin{aligned} X &= R\cos\delta\cos\alpha\,, & X' &= R\cos\delta'\cos\alpha', \\ Y &= R\cos\delta\sin\alpha\,, & Y' &= R\cos\delta'\sin\alpha', \\ Z &= R\sin\delta\,, & Z' &= R\sin\delta'. \end{aligned}$$

Through subtraction, in which we can neglect all but quantities of the first order with respect to the small differences $\alpha' - \alpha$ and $\delta' - \delta$, since the error which results in the values of X, Y and Z is less than half a unit in the seventh decimal place, we get

$$\begin{aligned} X - X' &= R\cos\delta'\sin\alpha'(\alpha' - \alpha) + R\sin\delta'\cos\alpha'(\delta' - \delta)\,, \\ Y - Y' &= -R\cos\delta'\cos\alpha'(\alpha' - \alpha) + R\sin\delta'\sin\alpha'(\delta' - \delta)\,, \\ Z - Z' &= -R\cos\delta'(\delta' - \delta)\,. \end{aligned}$$

But, from the well known formulas for the reduction of the fixed stars, we have

$$\alpha' - \alpha = aA + bB + E \quad \text{and} \quad \delta' - \delta = a'A + b'B\,,$$

in which

$$a = m + n \sin \alpha' \tan \delta', \quad a' = n \cos \alpha',$$
$$b = \cos \alpha' \tan \delta', \qquad b' = -\sin \alpha'.$$

Making these substitutions, we shall obtain

$$X - X' = (mY' + nZ')\,A + Y'E\,,$$
$$Y - Y' = -mX'A - Z'B - X'E\,,$$
$$Z - Z' = -nX'A + Y'B\,.$$

Since $mA + E$ is usually denoted by f, and we may write A' instead of $nA = g \cos G$ and $B = g \sin G$, our equations may be written

$$X - X' = fY' + A'Z',$$
$$Y - Y' = -fX' - BZ',$$
$$Z - Z' = -A'X' + BY'.$$

In most of the ephemerides f, log B and log A are given; then to the last add log of n expressed in seconds of arc; f, A' and B being thus expressed in seconds of arc, it will be most convenient to add to their logs the constant log 1.68557, whence the reductions above will be expressed in units of the seventh decimal place.

If it is required to reduce the coordinates to the equator and equinox of the beginning of a year previous to or following the current one, it is only necessary to increase, in the first case or diminish in the second, the value of A by the requisite number of units. This, however, must not be too large, otherwise the quantities of the second order may become sensible.

In computing the ephemeris of a planet, if we have not the mean coordinates but only the true coordinates of the sun, it will evidently be a saving of labor, to employ the formulas above to reduce the heliocentric coordinates of the planet from the mean to the true equinox and equator of date, and not those of the sun in the opposite direction.

MEMOIR No. 6.

Discussion of the Observations of the Great Comet of 1858, with the Object of Determining the Most Probable Orbit.

(Memoirs of the American Academy of Arts and Sciences, Vol. IX, pp. 67-100, 1867.)

Communicated by T. H. Safford, April 12, 1864.

The interesting physical aspect of this comet attracted to it, in an unusual degree, the attention of astronomers, a large part of whose energies were expended in obtaining observations for position. Consequently, we have a large mass of material for determining its orbit, not a little of which is of very good quality. Added to this, the long period of the apparition of the comet (nine months), would enable us to obtain the elements with considerable precision. Moreover, hints were thrown out that some other force besides gravity might affect its motion. Although these seem to have had no foundation other than the fact that the orbits derived from three normals did not well represent the intermediate observations, yet it is a matter of some interest to clear up the suspicion.

As the first step in the work, I determined to reduce the observations to uniformity, in respect to the places adopted for the comparison stars; which last I proposed to derive from all the material accessible to me. The desirableness of this course is evident when we consider that the observers at Bonn, Kremsmünster, Ann Arbor, and the two observatories in the southern hemisphere reobserved their comparison stars, in consequence of which their observations agree much better among themselves; while the rest contented themselves with places from Lalande, Bessel's Zones, or the British Association Catalogue, and their results exhibit larger probable errors. And as the comet was observed nearly simultaneously in Europe, the same comparison star was frequently used by a dozen observatories for the same night's work; and thus the stars of the latter class of observatories mentioned above are often found among those reobserved by the former. The result of this labor has convinced me that it has not been wasted; the good effect is apparent, particularly in the Liverpool and Göttingen observations.

A catalogue of all the stars used for comparison having been formed, the following authorities were consulted for material:

Baily's Lalande, Piazzi, Bessel's Zones (Weisse's Reduction), Struve Catalogus Generalis, Taylor, Rümker, Argelander's Southern Zones (Oeltzen), Robinson's Armagh Catalogue, Johnson's Radcliffe Catalogue, Greenwich Twelve Year and Six Year Catalogues, Mädler, Greenwich Observations, 1854–1860, Henderson Edinburgh Observations, Challis Cambridge Observations, Leverrier Paris Observations, 1856–59.

Leverrier commenced, in 1856, to reobserve the stars of Lalande; hence quite a number of the stars the observers had taken from this source, were found in the Paris Observations. The searching them out and reducing them entailed considerable labor. In addition to the material before mentioned, that furnished by the observatories at which the comparison stars were reobserved, was, of course, not omitted.

All this material was reduced to 1858.0, and to the standard of Wolfer's Tabulæ Reductionum, by applying the systematic corrections given by Auwers, in Astr. Nachr., No. 1300, with the modifications suggested by Mr. Safford, in No. 1368. The systematic corrections for Robinson are found in Astr. Nachr., No. 1408. Also, the following, kindly furnished by Mr. Safford, were employed:

	R. A.	DEC.
Greenwich Six Year Catalogue, . . .	$+0^s.017$	
Greenwich Observations, 1854-60, . . .	$+0.027$	$+0.''70$
Paris Observations, 1856-59, ...	$+0.056$	$+0.19$

In a few cases, mostly Piazzi stars, where the observations indicated proper motion, it was taken into account. With regard to the stars used in the southern observations, those common to the northern being excepted, they were retained without change, or when the same star had been used at both observatories, the observations were combined, allowing a weight of 3 to the Cape and of 2 to the Santiago observation. However, the place of the Santiago star, No. 57, equivalent to Cape No. 95, is wrong, seemingly an error of reduction; hence the Cape place has been adopted. And Santiago, No. 49, differing $7''.5$, in declination, from its equivalent, Cape No. 87, the Cape declination appearing the better, has been retained.

No.	α 1858.0 h m s	δ 1858.0 ° ′ ″	No.	α 1858.0 h m s	δ 1858.0 ° ′ ″
1	9 11 35.277	+25 0 59.98	54	10 44 8.277	+33 47 57.50
2	9 23 19.992	25 2 12.96	55	10 45 21.594	34 58 45.71
3	9 25 52.434	24 5 6.93	56	10 47 3.944	34 47 31.18
4	9 29 26.949	25 1 49.81	57	10 47 51.964	34 15 49.07
5	9 29 41.987	25 18 21.93	58	10 52 35.910	35 13 36.25
6	9 30 47.635	26 34 35.94	59	10 56 35.054	35 7 11.70
7	9 32 23.230	26 38 48.99	60	10 59 36.820	35 36 32.64
8	9 33 27.857	26 33 26.79	61	11 0 45.884	35 29 1.63
9	9 37 42.094	27 41 55.28	62	11 1 29.939	37 4 43.39
10	9 37 47.087	24 25 33.49	63	11 1 58.610	35 40 37.52
11	9 38 33.273	27 34 38.43	64	11 2 24.790	36 6 12.73
12	9 38 42.482	27 48 43.19	65	11 4 16.855	35 46 40.87
13	9 44 17.169	28 26 25.61	66	11 4 37.860	35 33 27.80
14	9 45 49.118	28 21 41.66	67	11 10 16.610	36 13 5.54
15	9 45 51.470	27 57 29.24	68	11 10 48.100	33 52 6.00
16	9 46 34.633	28 1 10.89	69	11 11 4.961	36 15 52.34
17	9 48 45.001	28 46 15.38	70	11 13 48.264	36 25 24.60
18	9 49 3.501	29 14 1.93	71	11 14 24.766	36 6 48.63
19	9 50 10.023	29 15 28.20	72	11 17 49.011	35 56 46.68
20	9 51 24.453	30 19 26.44	73	11 19 30.334	36 32 58.19
21	9 53 8.109	29 27 50.85	74	11 20 16.048	36 9 7.63
22	9 56 54.727	30 26 9.00	75	11 22 8.300	36 25 12.10
23	9 58 59.212	30 12 16.94	76	11 27 39.248	36 11 24.50
24	10 3 36.641	30 50 50.42	77	11 28 14.746	36 42 40.78
25	10 6 0.703	32 7 41.13	78	11 29 52.400	36 23 30.10
26	10 6 56.647	32 10 17.05	79	11 30 28.154	36 23 31.78
27	10 8 9.965	30 0 58.15	80	11 31 6.811	36 23 1.60
28	10 9 27.359	31 35 38.88	81	11 33 33.925	35 0 12.15
29	10 9 50	31 8 36	82	11 38 8.137	36 40 53.08
30	10 10 27.200	31 19 36.16	83	11 41 22.161	35 37 17.78
31	10 12 33.258	32 8 25.35	84	11 42 18.698	35 43 13.14
32	10 12 45.450	31 22 26.94	85	11 48 39.684	36 7 52.28
33	10 14 12.375	32 15 26.28	86	11 48 57.507	36 14 16.51
34	10 14 47.246	31 2 47.57	87	11 54 23.109	36 50 12.34
35	10 14 56.648	31 33 9.66	88	11 55 23.490	36 31 5.11
36	10 16 57.222	31 5 41.78	89	11 57 25.064	36 21 29.55
37	10 23 37.058	31 46 9.62	90	11 59 22.626	36 7 52.04
38	10 23 47.154	33 6 25.56	91	12 8 41	36 2
39	10 25 56.094	33 14 35.93	92	12 9 21.473	33 51 20.73
40	10 26 29.210	32 24 43.22	93	12 14 5.054	35 28 35.44
41	10 27 27.290	32 30 36.72	94	12 18 0.818	35 33 5.54
42	10 29 41.545	33 28 13.95	95	12 23 36.015	34 32 7.24
43	10 29 46.127	33 25 30.26	96	12 24 3.679	34 40 32.25
44	10 30 43.132	32 42 45.11	97	12 24 38.593	34 42 4.90
45	10 34 4.401	33 53 25.44	98	12 26 38.907	34 1 58.92
46	10 34 13.347	32 26 21.00	99	12 30 5.468	33 48 31.59
47	10 35	34 10	100	12 40 14.318	33 20 42.67
48	10 35 11.612	34 6 20.68	101	12 44 8.808	32 15 8.42
49	10 36 27.312	33 21 49.84	102	12 48 56.000	32 46 19.64
50	10 37 50.279	33 20 33.31	103	12 49 22.827	39 5 10.26
51	10 38 50.569	34 18 20.60	104	12 53 28.505	31 33 8.05
52	10 39 45.817	34 20 17.23	105	12 53 38.459	32 32 45.83
53	10 44 6.512	+32 7 11.86	106	12 55 34.619	+31 7 17.08

No.	α 1858.0	δ 1858.0
	h m s	° ′ ″
107	12 57 5.635	+31 31 16.16
108	12 57 16	31 14
109	12 57 26.035	30 58 58.19
110	12 59 23.612	29 47 28.59
111	13 0 21.817	28 23 16.41
112	13 2 21.623	31 0 9.03
113	13 2 45.467	31 11 36.59
114	13 7 53.393	30 9 19.48
115	13 9 5.077	30 5 55.53
116	13 10 14.299	29 47 44.00
117	13 12 20.788	29 18 25.90
118	13 18 20.109	24 35 44.78
119	13 20 10.842	26 59 50.88
120	13 21 46.769	28 5 9.80
121	13 22 2.800	29 11 20.02
122	13 23 8.620	28 24 36.87
123	13 23 45.303	28 23 16.90
124	13 25	28 20
125	13 30 3.869	26 36 19.12
126	13 33 22.650	26 38 49.62
127	13 37 33.182	26 0 5.60
128	13 40 7.651	26 24 59.35
129	13 44 19.729	24 20 51.41
130	13 45 56.310	24 15 58.17
131	13 46 12.072	24 2 8.33
132	13 46 46.651	24 51 40.80
133	13 51 39.354	24 38 30.49
134	13 51 59.705	22 23 26.35
135	13 54 25.217	22 39 58.50
136	13 55 20.693	22 14 33.72
137	14 7 56.650	19 9 59.50
138	14 9 11.160	19 55 24.82
139	14 9 23.644	19 34 29.31
140	14 11 14.667	19 6 2.59
141	14 13 2.053	16 57 35.06
142	14 17 27.790	16 55 11.16
143	14 20 0.953	17 3 22.25
144	14 21 31.064	16 45 49.85
145	14 23 11.387	16 50 40.76
146	14 28 12.905	13 43 16.55
147	14 33 46.108	13 52 14.60
148	14 33 55.070	14 8 48.84
149	14 34 22.174	14 20 23.45
150	14 34 54.307	12 16 29.77
151	14 39 4.926	13 42 18.62
152	14 42 33.357	10 38 27.29
153	14 42 47.985	10 47 39.30
154	14 44 10.403	10 18 35.81
155	14 44 35.229	10 35 48.49
156	14 51 54.337	7 10 15.57
157	14 57 3.620	6 3 17.78
158	14 58 1.431	7 15 39.93
159	14 58 12.582	+ 6 51 17.26

No.	α 1858.0	δ 1858.0
	h m s	° ′ ″
160	14 59 35.593	+ 6 19 28.56
161	14 59 57.698	6 54 50.52
162	15 0 33.375	6 49 9.03
163	15 4 21.645	3 22 6.91
164	15 5 11.046	7 10 34.03
165	15 8 54.017	6 59 39.72
166	15 12 35.554	+ 3 51 2.73
167	15 17 4.230	— 0 2 17.19
168	15 20 28.516	— 0 6 57.69
169	15 20 44.108	+ 0 23 21.61
170	15 23 56.400	— 0 14 16.79
171	15 30 20.818	3 7 57.60
172	15 33 46.943	3 31 59.42
173	15 37 0.458	— 3 23 9.11
174	15 37 16.575	+ 6 52 30.92
175	15 41 30.788	— 3 22 46.67
176	15 43 44.425	+ 4 54 29.08
177	15 44 11.680	— 7 36 47.93
178	15 44 33	6 53
179	15 46 54.770	7 40 54.50
180	15 52 4.738	6 53 37.22
181	15 52 26.783	6 42 53.62
182	15 53 7.954	8 0 23.10
183	15 55 1.103	10 13 57.47
184	15 56 33.959	10 58 40.88
185	16 0 21.617	13 22 56.05
186	16 0 41.310	9 42 57.87
187	16 2 59.572	14 0 27.35
188	16 3 6.618	13 36 59.00
189	16 4 24.098	13 22 3.31
190	16 4 41.570	10 6 50.01
191	16 5 42.980	12 40 2.27
192	16 5 59.266	16 22 13.69
193	16 6 12.084	13 37 42.35
194	16 6 29.082	10 3 1.35
195	16 6 59.660	14 16 29.96
196	16 8 32.770	13 17 23.70
197	16 10 3.544	13 5 23.42
198	16 11 34.281	16 8 19.19
199	16 14 44.967	16 40 51.94
200	16 20 9.964	15 53 23.81
201	16 23 0.959	16 17 57.48
202	16 23 43.537	21 9 29.66
203	16 30 18.736	18 32 9.72
204	16 34 32.419	21 29 43.20
205	16 34 36.256	21 4 1.58
206	16 37 12.274	18 52 12.21
207	16 40 7.990	21 41 3.72
208	16 41 6.008	24 23 11.15
209	16 41 7.071	21 35 54.42
210	16 41 50.606	24 15 51.12
211	16 52 37.309	26 25 39.09
212	16 53 3.366	—27 43 31.09

No.	α 1858.0 h m s	δ 1858.0 ° ′ ″
213	16 53 10.777	13 20 26.52
214	16 55 4.596	28 2 57.96
215	16 55 31.065	28 22 0.37
216	16 57 44.663	28 3 54.25
217	16 58 39.016	27 55 53.96
218	16 59 24.776	27 54 39.07
219	17 5 10.403	29 52 34.77
220	17 5 37.478	29 41 14.99
221	17 6 47.473	30 2 30.62
222	17 8 16.677	30 0 8.39
223	17 9 20.175	29 42 53.91
224	17 10 7.169	31 12 16.83
225	17 12 14.840	31 25 56.61
226	17 13 4.999	31 26 22.48
227	17 17 17.581	32 50 3.02
228	17 19 44.388	32 52 53.83
229	17 23 0.424	34 10 1.01
230	17 23 54.929	34 16 20.54
231	17 29 16.449	35 21 48.14
232	17 31 4.961	35 33 46.60
233	17 33 12.654	36 52 6.28
234	17 34 26.669	36 42 1.87
235	17 40 15.178	37 28 49.04
236	17 41 32.567	37 45 43.95
237	17 44 36.555	38 35 8.95
238	17 45 57.092	38 38 45.01
239	17 50 27.549	39 13 45.64
240	17 50 39.138	39 39 2.66
241	17 54 38.324	40 38 8.40
242	17 55 11.130	40 26 50.86
243	18 2 23.867	41 44 28.49
244	18 5 14.187	41 56 26.36
245	18 5 36.073	43 12 19.51
246	18 7 1.414	42 30 48.85
247	18 7 5.615	42 15 28.83
248	18 8 31.282	42 20 5.76
249	18 10 43.566	43 49 49.10
250	18 10 52.913	43 1 59.55
251	18 11 7.779	42 37 40.29
252	18 12 9.145	42 59 37.64
253	18 12 36.869	42 39
254	18 13 58.237	44 10 30.94
255	18 18 12.611	44 14 43.36
256	18 18 54.896	43 55 46.90
257	18 21 39.243	44 41 8.96
258	18 27 50.763	45 34 44.15
259	18 33 11.685	46 18 24.55
260	18 35 45.167	46 43 42.28
261	18 36 12.466	46 31 17.59
262	18 41 53.432	46 45 22.59
263	18 43 23.840	—47 26 22.74

No.	α 1858.0 h m s	δ 1858.0 ° ′ ″
264	18 44 10.419	—47 49 47.56
265	18 44 28.573	47 47 17.12
266	18 46 26.978	47 45 18.57
267	18 46 31.226	47 34 3.31
268	18 48 13.570	48 9 23.70
269	18 49 54.520	48 28 21.45
270	18 52 54.066	48 54 32.28
271	18 53 55.641	48 36 18.11
272	18 54 10.142	48 51 11.31
273	18 56 41.428	49 14 21.17
274	18 59 22.395	49 32 0.29
275	19 3 52.330	49 46 24.01
276	19 5 52.858	50 13 41.25
277	19 6 44.688	49 42 18.51
278	19 12 11.637	50 30 20.14
279	19 14 33.567	50 46 57.51
280	19 19 10.963	51 3 5.09
281	19 19 18.450	51 16 7.09
282	19 23 22.183	50 51 50.34
283	19 23 34.615	51 34 45.35
284	19 26 53.202	51 45 7.15
285	19 29 44.020	51 51 59.82
286	19 30 15.827	51 50 49.35
287	19 30 35.565	52 5 43.17
288	19 33 0.830	52 8 8.42
289	19 33 15.841	52 16 22.64
290	19 34 26.520	52 21 40.38
291	19 38 11.319	52 25 21.16
292	19 39 33.763	52 35 4.72
293	19 40 57.485	52 47 37.75
294	19 42 1.927	52 40 19.22
295	19 45 4.527	53 10 20.41
296	19 45 36.756	53 4 53.37
297	19 50 33.215	53 21 50.75
298	19 50 43.777	53 12 39.53
299	19 56 47.461	53 30 37.25
300	19 57 18.613	52 58 52.81
301	20 0 15.512	53 45 3.25
302	20 2 33.653	54 1 31.30
303	20 5 17.695	54 11 0.36
304	20 6 48.770	54 14 53.47
305	20 9 15.916	54 29 49.18
306	20 11 41.803	54 42 28.61
307	20 15 45.780	54 16 41.37
308	20 16 25.769	54 39 2.12
309	20 17 13.505	54 45 46.37
310	20 18 46.916	55 33 8.81
311	20 19 7.030	55 2 3.04
312	20 21 57.156	54 59 26.98
313	20 22 58.968	—54 56 2.85

No.	α 1859.0	δ 1859.0	No.	α 1859.0	δ 1859.0
	h m s	° ′ ″		h m s	° ′ ″
314	20 25 26.390	—55 3 20.79	339	21 23 0.430	—58 0 17.76
315	20 27 7.650	55 18 29.95	340	21 23 22.890	57 42 4.33
316	20 27 13.290	55 24 33.43	341	21 25 2.280	58 0 4.65
317	20 31 23.834	55 36 23.14	342	21 28 35.340	58 20 30.78
318	20 33 40.930	55 36 2.15	343	21 29 54.810	58 4 24.05
319	20 34 34.990	55 41 47.50	344	21 30 52.172	58 22 23.85
320	20 38 21.560	55 43 23.42	345	21 32 9.160	58 15 1.52
321	20 39 58.730	55 53 23.14	346	21 33 18.394	58 32 14.83
322	20 43 12.100	56 6 49.19	347	21 33 39.750	58 0 28.20
323	20 44 29.930	55 59 27.04	348	21 33 57.470	57 55 21.35
324	20 45 35.640	55 45 12.47	349	21 35 8.380	58 41 34.64
325	20 47 1.230	56 14 47.53	350	21 37 51.400	58 40
326	20 47 55.460	56 20 9.53	351	21 40 18.106	58 57 17.50
327	21 1 2.480	57 5 12.43	352	22 8 41.970	60 32 20.95
328	21 2 14.520	57 5 6.31	353	22 8 48.310	60 57 36.81
329	21 4 50.880	57 8 13.18	354	22 9 36.790	60 49 14.47
330	21 8 3.260	57 18 3.90	355	22 11 13	61 8 11.00
331	21 10 45.750	57 12 15.77	356	22 12 6.080	60 39 16.90
332	21 11 6.720	57 26 34.46	357	22 16 36.870	61 5 50.15
333	21 12 39.970	57 23 55.34	358	22 18 40.420	61 17 31.70
334	21 14 20.600	57 51 22.09	359	22 21 12.300	61 13 40.79
335	21 18 19.350	57 45 21.06	360	22 23 53.980	61 32 27.09
336	21 20 26.300	57 29 5.95	361	22 25 40.970	61 40 32.46
337	21 20 48.940	57 46 28.59	362	22 27 25.610	61 43 53.15
338	21 21 55.210	—57 55 14.83	363	22 30 54.250	—61 57 58.99

The following are the authorities for the observations and the places of the comparison stars:

ALTONA. Astr. Nachr., L. 187.

ANN ARBOR. Astr. Nachr., XLIX. 179. Brünnow's Astr. Notices, I. 6, 53.

ARMAGH. Monthly Notices, XIX. 305.

BATAVIA. Astr. Nachr., L. 107.

BERLIN. Astr. Nachr., XLVIII. 333, LI. 65.

BONN. Astr. Nachr., XLIX. 253, LI. 187.

BRESLAU. Astr. Nachr., L. 37.

CAMBRIDGE, ENG. Astr. Nachr., L. 243.

CAMBRIDGE, U. S. Astr. Nachr., LI. 273. Brünnow's Astr. Notices, I. 71.

CAPE OF GOOD HOPE. Mem. Astr. Soc., XXIX. 59–83. The observations were made with two different instruments; those made with the larger have been denoted in the list of observations which follows by "Cape 1," and those made with the smaller by "Cape 2."

CHRISTIANIA. Astr. Nachr., LII. 277.

COPENHAGEN. Oversigt kgl. danske Videnskabernes Selskabs, 1858.

DORPAT. Beob. Kaiserl. Sternw. Dorpat, Vol. XV. These observations are published in a crude form, and I was unable to reduce and use them, from a want of the instrumental constants.

DURHAM. Astr. Nachr., L. 11.

FLORENCE. Astr. Nachr., XLVIII. 347, 355, XLIX. 57, L. 97. The observation of October 13 is erroneous as regards the comparison star, which it seems should be Piazzi XV. 227.

GENEVA. Astr. Nachr., XLIX, 115, L. 21.

GÖTTINGEN. Astr. Nachr., XLIX. 235, L. 11.

GREENWICH. Greenwich Observations for 1858. Monthly Notices, XIX. 12.

KÖNIGSBERG. Astr. Nachr., L. 71, LIII. 289.

KREMSMÜNSTER. Astr. Nachr., XLIX. 68, 79, 257, LI. 23.

LEYDEN. Astr. Nachr., L. 157. The observer is mistaken in the comparison star of his last observation ; it should be Weisse, XV. 369.

LIVERPOOL. Astr. Nachr., XLIX. 267. Monthly Notices, XIX. 54.

MARKREE. Observations on Donati's Comet, 1858, at Markree.

PADUA. Astr. Nachr., XLVIII. 357.

PARIS. Annales de l'Observatoire Imperial, Paris. Tome XIV. Observations.

PULKOVA. Astr. Nachr. L. 307. Beobachtungen der Grossen Cometen 1858. Otto Struve.

SANTIAGO. Astr. Nachr., LIII. 131. Astr. Jour., VI. 100.

VIENNA. Astr. Nachr., XLVIII. 349, XLIX. 43, 53, L. 227, LII. 57.

WILLIAMSTOWN. Astr. Nachr., L. 7. As the latitude and longitude of the place are uncertain, I have not reduced these observations.

WASHINGTON. Astr. Nachr., XLIX. 55, 113, 363. Astr. Jour., V. 150, 158, 166, 180. The comparison star of October 1 is mistaken.

The typographical errors to be met with are so numerous I cannot undertake to mention them. To render the reduction of the comparison stars from mean to apparent place uniform, the elements of reduction in the British Nautical Almanac for 1858 were adopted as the standard ; and the same will be used in reducing our normals from apparent to mean places. Consequently, it becomes necessary to add to the observations in which the elements of the Berlin Jahrbuch were used, quantities easily obtained from this small ephemeris.

	R. A.	DEC.		R. A.	DEC.
June 15	$+0''.09$	$+0''.18$	Sept. 18	$+0''.08$	$+0''.03$
July 15	+0.02	+0.22	Oct. 3	+0.07	−0.04
Aug. 14	+0.03	+0.18	Oct. 18	+0.04	−0.19
Sept. 3	+0.05	+0.10	Nov. 2	+0.14	−0.23

For the reduction of the observations for parallax, and the computation of the perturbations, and for comparison, an ephemeris was computed from these elements published by Searle in the Astronomical Journal, V. 188, Searle's own ephemeris not being sufficiently exact for the purpose of comparison.

$$T = \text{Sept. } 29.75230 \quad 1858 \text{ Washington Mean Time}$$

$$\left.\begin{aligned} \pi - \Omega &= 129^\circ\ 6'\ 24''.8 \\ \Omega &= 165\ 18\ 46.2 \\ i &= 116\ 57\ 46.1 \end{aligned}\right\} \text{Mean Equinox and Ecliptic 1858.0}$$

$$\varphi = 85\ 21\ 21.2$$

$$\log q = 9.7622362$$

In the following list the observations of the comet are given reduced for parallax, and are made to accord with the places of the comparison stars given in the foregoing catalogue. Gould's list of Longitudes (in the American Ephemeris) has been used in getting the Paris M. T. of Observation. The comparisons in the last two columns are Obs. — Cal. The declinations of the southern observations have generally been reduced to the time of observing the right ascension; that observation of right ascension being selected which was nearest in time and which had the same comparison star.

Paris M. T. of Observation	Place of Observation	α	δ	Number of Comp. Star	$\Delta\alpha$	$\Delta\delta$
		° ′ ″	° ′ ″		″	″
June 7.41071	Florence	141 14 47.79	+24 21 54.73	3	+21.69	+ 6.26
8.37659	"	141 15 36.99	24 27 52.30	10	+39.17	—15.66
9.42802	"	141 16 20.54	24 34 48.42	10	+27.71	— 7.33
10.39044	"	141 17 25.48	24 41 10.00	10	+23.44	+ 5.67
11.40973	"	141 19 3.43	24 47 35.12	1	+28.89	+ 5.05
12.37591	Padua	141 20 31.82	24 53 36.67	4	+11.55	+ 5.14
12.41803	Florence	141 20 21.71	24 53 56.68	1	— 3.35	+10.15
13.37729	Padua	141 22 34.98	24 59 27.99	4	+ 6.81	—14.10
13.40557	Florence	141 22 16.33	25 0 14.83	1	—15.72	+22.34
13.43268	Berlin	141 22 43.08	24 59 50.15	2	+ 7.30	—12.31
14.41069	"	141 24 58.54	25 5 52.65	2	— 0.89	— 7.69
14.41609	Vienna	141 25 15.40	25 5 55.69	2–5	+15.13	— 6.61
15.39007	Florence	141 28 20.08	25 11 23.30	5	+37.49	—31.86
15.40675	Vienna	141 27 58.18		2–5	+14.65	
15.44201	Berlin	141 27 36.29	25 12 2.13	5	— 9.96	— 4.80
16.39944	Kremsmünster	141 30 54.96	25 17 48.71	5	+10.41	— 8.58
16.41628	Berlin	141 30 39.41	25 17 49.11	5	— 8.36	—14.19
17.39261	Florence	141 34 31.47	25 23 26.85	5	+28.40	—23.47
19.37441	"	141 41 42.14	25 35 32.96	5	+12.05	+ 7.07
19.38451	Padua	141 42 8.91	25 35 39.64	5	+23.63	+10.24
28.38292	Florence	142 29 25.62	26 26 8.67	8	+26.28	— 6.86
28.61976	Cambridge, U. S.	142 30 24.43	26 27 36.77	6	— 7.25	+ 2.80
29.38224	Florence	142 35 56.97	26 31 43.74	8	+22.16	— 2.41
29.41947	Berlin	142 36 2.15	26 31 52.88	6	+13.31	— 5.57
30.37599	Florence	142 42 46.34	26 37 8.61	8	+22.98	— 5.44
30.38577	Vienna	142 42 24.72	26 37 20.26	8	— 2.72	+ 2.98
July 2.37816	Florence	142 56 57.17	26 48 14.56	7	+ 4.77	+ 1.86
8.38159	"	143 46 55.06	27 20 55.54	11	+34.65	— 9.74
9.38324	Vienna	143 55 34.04	27 26 44.86	12	+ 6.24	+ 6.85
9.60789	Washington	143 57 42.00	27 27 56.51	11	+ 9.40	+ 6.19
10.37333	Florence	144 5 0.20	27 32 2.33	11	+16.80	— 1.59
10.59343	Washington	144 6 50.62	27 33 16.24	9	+ 1.75	— 0.72
10.59343	"	144 6 59.45	27 33 18.41	11	+10.58	+ 1.45
11.59576	"	144 16 28.58	27 38 51.84	9	— 0.61	+ 1.42
12.37144	Florence	144 24 2.74	27 43 13.41	9	— 5.59	+ 3.93
13.37158	"	144 34 23.43	27 48 47.14	12	+10.19	+ 2.20
13.59089	Cambridge, U. S.	144 36 26.15	27 50 6.11	12	— 1.67	+ 7.56
14.36879	Florence	144 44 34.08	27 54 18.74	12	+ 3.24	— 2.50
14.58534	Washington	144 46 51.57	27 55 38.77	9	+ 4.72	+ 4.34
15.58781	Cambridge, U. S.	144 57 23.93	+28 1 20.09	15	— 1.44	+ 4.97

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
July 15.58781	Cambridge, U. S.	144 57 25.18	+28 1 17.14	16	— 0.19	+ 2.02
15.58803	Washington	144 57 30.05	28 1 17.20	12	+ 4.55	+ 2.00
16.58028	"	145 8 17.33	28 6 59.62	12	+ 5.49	+ 5.02
17.58135	"	145 19 23.15	28 12 55.11	14	+ 4.72	+15.60
19.36496	Florence	145 40 9.14	28 22 35.10	14	+26.92	—25.84
19.57128	Cambridge, U. S.	145 42 5.95	28 24 25.03	13	— 0.82	+11.59
19.57128	"	145 42 8.63	28 24 14.88	14	+ 1.86	+ 1.44
20.35855	Florence	145 51 27.80	28 28 46.13	13	+ 4.39	— 5.81
21.58017	Washington	146 6 8.33	28 36 7.34	14	+ 1.28	+ 0.80
23.61560	Ann Arbor	146 31 25.02	28 48 22.98	17	— 2.55	— 0.64
24.58362	Washington	146 43 53.23	28 54 25.65	17	+ 0.71	+ 5.84
25.57816	"	146 56 54.02	29 0 37.18	17	+ 0.72	+ 7.35
27.57886	"	147 23 48.85	29 13 12.53	18	— 1.71	+ 5.10
28.57465	"	147 37 32.23	29 19 43.76	19	— 7.18	+12.22
29.57953	"	147 51 52.66	29 25 49.46	21	+ 0.36	—14.60
31.35674	Florence	148 17 13.40	29 37 38.77	21	—28.85	—12.75
Aug. 4.35187	"	149 19 6.35	30 5 14.06	27	— 3.54	—14.07
4.37075	Berlin	149 19 22.65	30 5 46.12	23	— 5.38	+ 9.91
4.57000	Washington	149 22 39.36	30 7 9.00	23	— 0.37	+ 7.56
5.34162	Kremsmünster	149 35 2.22	30 12 41.42	23	— 8.19	+ 7.34
5.34827	Florence	149 35 27.94	30 13 3.26	23	+12.02	+26.29
5.54365	Cambridge, U. S.	149 38 31.44	30 14 9.93	20	+ 3.83	+ 8.10
5.54365	"	149 38 21.08	30 14 8.53	23	— 6.53	+ 6.70
6.34048	Florence	149 51 27.72	30 19 54.03	23	— 9.48	+ 4.37
7.36200	Berlin	150 8 43.81	30 27 30.63	22	— 4.35	+ 6.21
7.56564	Washington	150 12 8.39	30 28 59.20	23	— 7.85	+ 3.44
8.56004	"	150 29 24.36	30 36 40.41	24	— 0.50	+ 9.71
10.33796	Kremsmünster	151 1 3.61	30 50 22.10	24	+ 6.28	+11.63
10.35184	Berlin	151 1 5.40	30 50 24.07	24	— 6.98	+ 7.07
10.56020	Washington	151 4 50.17	30 52 5.23	24	— 9.79	+12.49
11.33986	Kremsmünster	151 19 15.59	30 58 12.55	24	+ 0.36	+ 6.60
12.33952	"	151 37 33.32	31 6 17.22	36	—20.96	+ 8.66
12.59010	Ann Arbor	151 42 30.69	31 8 5.47	34	— 7.89	— 5.36
13.58572	"	152 1 33.00	31 16 24.08	30–2	—10.57	+ 2.20
14.33189	Vienna	152 16 14.43	31 22 42.03	32	— 4.01	+ 6.51
14.34131	Kremsmünster	152 16 19.63	31 22 45.35	32	— 9.94	+ 5.09
14.37376	Copenhagen	152 17 0.51	31 23 22.17	32	— 7.45	+25.54
14.571921	Ann Arbor	152 21 1.85	31 24 37.64	32	— 1.10	+ 0.91
15.556670	Washington	152 40 53.49	31 32 59.51	35	+ 6.95	+ 0.20
15.578418	Ann Arbor	152 41 13.22	31 33 18.56	35	+ 0.24	+ 8.05
16.329858	Florence	152 56 21.66	31 39 25.74	35	—13.09	—14.24
16.367755	Copenhagen	152 57 18.67	31 40 24.83	32	— 3.02	+25.08
16.550079	Washington	153 1 12.45	31 41 39.36	35	+ 4.49	+ 4.30
17.327305	Vienna	153 17 15.93	31 48 22.98	25–6	— 7.69	— 1.78
17.335186	Kremsmünster	153 17 29.01	31 48 33.89	28	— 4.56	+ 4.95
17.360620	Copenhagen	153 18 35.17	31 48 40.92	37	+29.33	— 1.52
17.375546	"	153 19 4.92	31 48 50.15		+40.15	— 0.23
17.541171	Washington	153 22 1.02	31 50 19.63	35	+ 5.74	+ 1.18
17.568303	Ann Arbor	153 22 29.67	31 50 35.82	37	— 0.15	+ 2.92
18.320072	Vienna	153 38 49.51	31 57 23.25	35	+12.98	+ 7.12
19.346349	Berlin	154 1 6.13	32 6 48.19	31	+ 0.94	+13.47
19.376823	Cambridge, Eng.	154 1 46.76	+32 6 38.02	46	+ 0.80	—13.45

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Aug. 19.544957	Washington	154 5 37.10	+32 8 36.44	26	+ 6.69	+12.53
19.548659	Cambridge, U. S.	154 5 27.88	32 8 29.71	53	— 7.49	+ 3.71
20.544643	"	154 28 57.08	32 17 51.52	33	— 8.00	+12.26
20.546599	"	154 28 54.87	32 17 43.99	40	—12.90	+ 3.64
21.330193	Kremsmünster	154 46 5.90	32 25 20.05	41	— 8.55	+18.06
22.539509	Washington	155 14 45.23	32 36 45.15	41	—11.54	+10.71
22.577310	Ann Arbor	155 15 44.05	32 37 3.12	41	— 7.46	+ 6.82
23.341254	Königsberg	155 34 14.71	32 44 9.54	38	—15.77	—11.18
23.360472	Copenhagen	155 34 37.11	32 44 21.26	38	—21.84	—10.66
23.370056	Cambridge, Eng.	155 35 12.50	32 44 42.35	38	— 0.64	+ 4.82
23.379349	"	155 35 21.95	32 44 52.87	44	— 4.97	+ 9.89
23.545062	Washington	155 39 21.99	32 46 28.14	44	—11.08	+ 8.04
23.563000	Ann Arbor	155 39 52.38	32 46 40.21	44	— 7.40	+ 9.59
24.328091	Königsberg	155 59 7.10	32 54 9.92	38	— 5.00	+ 7.73
24.333372	Copenhagen	155 59 21.07	32 54 19.14	38	+ 0.92	+13.82
24.335916	"	155 59 27.58	32 54 7.43		+ 3.55	+ 0.60
24.538632	Washington	156 4 26.18	32 56 20.54	38	— 7.54	+13.21
25.304874	Vienna	156 24 20.98	33 4 0.52	38	— 0.15	+14.69
25.320927	"	156 24 28.28	33 3 58.26	38	—18.03	+ 2.75
25.381750	Cambridge, Eng.	156 26 0.38	33 4 36.78	38	—21.40	+ 4.70
25.536968	Washington	156 30 19.48	33 6 21.39	38	— 6.71	+15.75
26.374066	Christiania	156 52 43.15	33 14 47.88	44	— 1.30	+14.43
26.378525	"	156 52 32.18	33 14 43.05	39	—19.49	+ 6.88
26.395693	"	156 53 13.79	33 14 54.46	38	— 5.70	+ 7.79
26.482289	"	156 55 36.26	33 15 48.95	38	— 3.68	+ 9.47
27.370067	Cambridge, Eng.	157 19 59.28	33 24 58.51	43	— 3.51	+14.01
27.380735	Christiania	157 20 16.24	33 25 3.85	43	— 4.31	+12.76
28.309822	Vienna	157 46 20.57	33 34 32.47	38	—16.50	+ 5.07
28.318270	Berlin	157 46 39.71	33 34 46.46	42	—11.92	+13.79
28.319291	Geneva	157 46 50.62	33 34 45.89	49	— 2.77	+12.59
28.322451	"	157 47 4.49	33 34 49.00	50	+ 5.65	+13.73
28.480204	Christiania	157 51 24.47	33 36 33.58	38	— 6.96	+19.89
30.299305	Kremsmünster	158 45 36.15	33 55 39.06	48	— 6.40	+20.21
30.304230	Vienna	158 45 47.70	33 55 36.62	51–2	— 3.95	+14.75
30.309523	Florence	158 45 37.69	33 55 25.96	54	—23.73	+ 0.73
30.377443	Cambridge, Eng.	158 48 3.62	33 56 17.78	45	— 3.26	+ 9.48
30.526796	Cambridge, U. S.	158 52 31.74	33 57 54.46	48	—12.05	+11.38
31.291317	Vienna	159 16 32.49		51–2	—12.39	
31.319005	Kremsmünster	159 17 33.52	34 6 20.04	48	— 4.31	+12.83
31.337849	Copenhagen	159 17 59.57	34 6 25.76	48	—14.33	+ 6.54
31.339817	"	159 18 2.98	34 6 36.56		—14.69	+16.08
31.551986	Ann Arbor	159 24 53.06	34 8 48.07	57	—12.42	+12.21
Sept. 1.295680	Florence	159 49 5.83	34 16 36.92	57	—14.72	+ 5.70
1.309567	Kremsmünster	159 49 37.11	34 16 54.23	51	—11.00	+14.08
1.320362	Bonn	159 50 1.29	34 17 0.04	51	— 8.25	+13.02
1.322480	Christiania	159 50 3.13	34 17 0.94	51	—10.62	+12.57
1.326603	Berlin	159 50 11.89	34 17 6.08	51	— 9.75	+15.17
1.540701	Ann Arbor	159 57 17.76	34 19 16.27	57	—10.16	+ 8.18
1.563647	"	159 58 12.94	34 19 38.10	51	— 1.96	+15.32
2.298887	Kremsmünster	160 23 1.61	34 27 23.80	51	— 7.84	+ 9.99
2.301560	Florence	160 22 49.02	34 27 29.68	57	—25.94	+14.16
2.305161	Vienna	160 23 9.90	+34 27 32.33	56	—12.48	+14.50

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Sept. 2.327381	Geneva	160 24 1.89	+34 27 48.29	57	— 6.35	+16.22
2.337187	Copenhagen	160 24 19.82	34 27 50.99	51	— 8.67	+12.64
2.350639	Königsberg	160 24 51.38	34 27 46.76	55	— 4.90	— 0.21
2.406874	Christiania	160 26 42.59	34 28 19.63	51	—10.03	— 3.37
2.418651	Pulkova	160 27 12.15	34 28 48.55		— 4.86	+18.00
2.427687	Christiania	160 27 32.91	34 28 48.24	57	— 2.83	+11.90
3.281247	Vienna	160 57 29.49	34 37 56.89	55	— 5.72	+14.13
3.292555	Florence	160 58 0.01	34 38 2.05	56	+ 0.58	+12.07
3.295584	Vienna	160 57 46.70	34 37 59.35	55	—19.27	+ 7.42
3.322590	Geneva	160 58 56.61	34 38 20.04	56	— 7.34	+10.85
3.530240	Washington	161 6 26.13	34 40 36.24	55	— 5.76	+14.37
4.276201	Kremsmünster	161 33 42.46	34 48 33.42	56	—10.63	+16.17
4.289568	"	161 34 16.06	34 48 39.87	56	— 6.91	+14.13
4.300379	Florence	161 34 36.28	34 48 49.44	56	—10.86	+16.83
4.308592	Christiania	161 34 57.50	34 48 48.59	56	— 8.02	+10.76
4.308932	Geneva	161 35 4.38	34 48 56.46	56	— 1.90	+18.41
4.311267	Berlin	161 35 0.39	34 48 51.87	56	—11.11	+12.34
4.316907	Geneva		34 48 52.29	55		+10.87
4.321872	"	161 35 25.19		55	—10.05	
4.421240	Christiania	161 39 10.86	34 49 59.11	56	— 7.32	+ 9.72
5.293191	Florence	162 12 25.51	34 59 17.57	55	— 9.81	+17.00
5.383408	Armagh	162 15 54.08	35 0 15.53	55	—12.20	+18.25
5.419101	Christiania	162 17 20.59	35 0 30.13	56	— 9.39	+10.39
5.419101	"	162 17 23.46	35 0 35.01	55	— 6.52	+15.27
5.529968	Washington	162 21 43.11	35 1 40.76	56	— 7.61	+11.49
5.537181	Ann Arbor	162 21 58.91	35 1 48.01	55	— 8.84	+14.21
5.654786	Durham	162 26 34.00	35 3 1.50	55	—11.85	+14.05
6.329700	Copenhagen	162 53 33.17	35 10 10.64		—17.06	+23.08
6.350846	"	162 54 8.16	35 10 6.70	58	—33.77	+ 6.06
6.364638	Armagh	162 55 13.80	35 10 26.90	55	— 1.88	+17.74
6.524524	Washington	163 1 26.35	35 12 8.28	56	—21.99	+20.40
6.544780	Ann Arbor	163 2 31.25	35 12 14.35	58	— 7.04	+13.99
6.917566	"	163 18 0.35	35 16 2.85	58	— 5.65	+13.70
7.315249	Berlin	163 34 46.30	35 20 3.87	58	— 6.76	+12.91
7.367969	Cambridge, Eng.	163 37 1.79	35 20 42.31	59	— 6.24	+19.48
7.514951	Washington	163 43 31.43	35 21 52.33	61	+ 5.57	+ 0.87
8.322259	Copenhagen	164 18 58.21	35 30 3.29		+ 9.08	+12.00
8.340315	Königsberg	164 19 28.57	35 30 10.23	60	— 8.99	+ 8.36
8.516888	Cambridge, U. S.	164 27 29.44	35 31 53.17	60	— 3.92	+ 8.14
8.516888	"	164 27 21.05	35 31 46.46	61	—12.31	+ 1.43
8.516888	"	164 27 25.09	35 31 57.43	66	— 8.27	+12.40
9.299974	Geneva	165 3 26.88	35 39 26.56	66	— 6.95	+12.82
9.302115	Florence	165 3 40.97	35 39 16.42	63	+ 1.10	+ 1.48
9.312209	Bonn	165 3 57.18	35 39 30.23	60	—11.08	+ 9.61
9.315659	Königsberg	165 4 9.73	35 39 38.55	60	— 8.24	+15.98
9.315932	Berlin	165 4 7.88	35 39 21.79	60	—10.86	— 0.93
9.319178	Paris	165 4 14.60	35 39 37.57	66	—13.28	+13.02
9.523602	Washington	165 13 57.13	35 41 33.91	61	— 9.16	+14.92
10.263712	Kremsmünster	165 49 36.73	35 48 19.58	65	—14.43	+16.55
10.277018	Vienna	165 50 23.97	35 48 24.58	65–6	— 6.50	+14.44
10.284499	Florence	165 50 43.39	35 48 29.91	65	— 9.20	+15.78
10.294314	Kremsmünster	165 51 7.95	+35 48 35.62	65	—13.65	+16.26

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Sept. 10.306418	Berlin	165 51 50.46	+35 48 42.52	65	— 6.96	+16.70
10.307120	Copenhagen	165 52 3.80	35 48 36.08		+ 4.30	+ 9.89
10.325063	Königsberg	165 52 42.14	35 48 50.29	65	—10.49	+14.54
10.325709	Paris	165 52 32.51	35 48 45.94	65	—22.03	+ 9.85
10.354888	Armagh	165 54 12.98	35 49 1.87	65	— 8.07	+10.27
10.520362	Washington	166 2 38.05	35 50 44.35	65	+ 4.00	+25.32
10.628708	Bonn	166 7 46.05	35 51 32.36	65	—13.14	+16.62
11.280961	Kremsmünster	166 41 6.09	35 57 2.34	64	— 9.66	+14.46
11.301907	Geneva	166 42 6.50	35 57 16.88	65	—14.52	+17.61
11.319423	Paris	166 43 10.02	35 57 29.26	69	— 5.64	+22.33
11.321783	Copenhagen	166 43 13.97	35 57 16.04	65	— 9.05	+ 7.94
11.325972	Königsberg	166 43 29.17	35 57 29.14	65	— 6.92	+18.96
11.411657	Pulkova	166 47 59.70	35 58 6.22		— 4.47	+13.85
12.261121	Kremsmünster	167 33 18.94	36 4 47.28	69	—10.38	+14.42
12.286131	Christiania	167 34 43.59	36 5 1.27	68	— 7.88	+17.13
12.286559	Florence	167 34 44.73	36 4 53.86	67	— 8.15	+ 9.55
12.293542	Königsberg	167 34 53.84	36 5 9.51	69	— 7.21	+22.06
12.295472	Geneva	167 35 12.02	36 4 58.98	69	—10.15	+10.66
12.306918	"	167 35 50.51	36 5 3.35	71	— 9.32	+ 9.89
12.311009	Liverpool	167 35 58.70	36 5 11.76	62	—14.60	+16.46
12.321429	"	167 36 36.73	36 5 14.60	62	—10.88	+14.63
12.331857	"	167 37 12.38	36 5 20.44	62	— 9.58	+15.80
12.346679	Paris	167 38 6.03	36 5 28.54	69	— 4.79	+17.28
12.354098	Armagh	167 38 15.70	36 5 30.97	69	—19.59	+16.39
12.411405	Pulkova	167 41 25.95	36 5 53.48		—18.72	+13.41
12.519159	Washington	167 47 33.62	36 6 47.30	69	—11.68	+19.75
12.526442	Ann Arbor	167 47 58.52	36 6 45.74	69	— 8.03	+15.00
12.536618	"	167 48 33.25	36 6 48.84	67	— 7.21	+13.66
12.612848	Bonn	167 52 41.84	36 7 22.26	69	—13.13	+13.93
13.277211	Kremsmünster	168 30 27.71	36 11 58.86	69	—12.00	+15.48
13.277425	Vienna	168 30 32.55	36 11 56.74	69	— 7.90	+13.28
13.291278	Königsberg	168 31 19.32	36 12 1.52	71	— 9.26	+12.59
13.291733	Christiania	168 31 15.14	36 12 2.23	62	—15.02	+13.12
13.294974	Geneva	168 31 27.37	36 12 6.30	69	—14.06	+15.92
13.305238	Berlin	168 31 59.11	36 12 8.35	69	—18.00	+13.93
13.317599	Paris	168 32 50.64	36 12 18.72	69	— 9.48	+19.45
13.328567	Leyden	168 33 27.24	36 12 15.34	70	—11.07	+11.78
13.329885	Copenhagen	168 33 27.66	36 12 12.04		—15.24	+ 7.96
13.333017	Christiania	168 33 44.19	36 12 20.78	68	— 9.62	+15.47
13.335091	Cambridge, Eng.	168 33 50.03	36 12 22.04	71	—11.05	+15.93
13.371491	Leyden	168 35 52.93	36 12 37.07	69	—15.03	+16.77
13.516267	Washington	168 44 23.18	36 13 30.19	69	—12.22	+14.31
13.525041	Ann Arbor	168 44 54.92	36 13 33.94	69	—11.36	+14.72
14.273293	Kremsmünster	169 29 51.01	36 17 57.33	69	— 6.96	+13.63
14.288013	Königsberg	169 30 44.22	36 18 2.07	72	— 7.87	+13.57
14.295328	Vienna	169 31 1.74	36 18 3.61	67–9	—17.26	+12.73
14.307231	Geneva	169 31 50.02	36 18 7.03	69	— 9.23	+12.28
14.307231	"	169 31 48.81	36 18 5.89	70	—10.44	+11.14
14.316540	Leyden	169 32 34.48		74	— 2.62	
14.318520	Copenhagen	169 32 27.58	36 18 13.68		—16.83	+15.27
14.323235	Paris	169 32 52.48	36 18 17.61	69	— 9.41	+17.69
15.294684	Vienna	170 34 17.06	+36 22 47.86	72	— 8.36	+11.71

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Sept. 15.299126	Königsberg	170 34 31.52	+36 22 58.52	74	—11.24	+21.30
15.299256	Geneva	170 34 26.31	36 22 49.14	69	—16.96	+10.88
15.299256	"	170 34 26.10	36 22 49.20	70	—17.17	+10.94
15.308400	"	170 35 3.64	36 22 43.84	73	—15.32	+ 4.37
15.316643	Liverpool	170 35 38.37	36 22 56.37	81	—13.78	+14.90
15.324451	Berlin	170 36 3.78	36 22 59.39	75	—17.88	+16.04
15.332976	Cambridge, Eng.	170 36 43.46	36 22 59.71	74	—11.53	+14.31
15.337503	Leyden	170 36 58.28	36 22 56.14	73	—14.41	+ 9.66
15.330548	Liverpool	170 36 37.19	36 23 0.77	81	— 8.31	+15.96
15.340177	Cambridge, Eng.	170 37 7.72	36 22 59.88	75	—15.43	+12.76
15.344453	Liverpool	170 37 35.46	36 23 5.47	81	— 4.41	+17.33
16.282016	Kremsmünster	171 40 22.08	36 26 5.74	80	—10.77	+15.72
16.289858	Florence	171 40 42.92	36 26 10.11	78	— 3.41	+18.94
16.294017	Christiania	171 41 9.40	36 26 6.29	62	—13.14	+14.51
16.305316	Königsberg	171 41 56.83	36 26 3.94	76	—12.52	+10.53
16.349513	Leyden	171 45 3.20	36 25 58.38	76	— 9.59	— 1.30
16.354993	Cambridge, Eng.	171 45 28.16	36 26 17.25	76	— 7.41	+16.81
16.366305	Leyden	171 46 4.63	36 26 15.05	77	—17.98	+13.03
16.411922	Pulkova	171 49 22.50	36 26 19.33		—10.14	+11.15
16.527175	Ann Arbor	171 57 20.10	36 26 36.40	78	—14.97	+13.66
16.548921	"	171 58 52.17	36 26 40.60	80	—14.29	+15.28
16.638835	Bonn	172 5 18.22	36 26 47.11	76	— 7.51	+11.66
17.259056	Vienna	172 49 50.32	36 27 36.18	82	— 8.84	+16.79
17.264848	Kremsmünster	172 50 9.59	36 27 36.31	82	—15.02	+16.64
17.282358	Königsberg	172 51 27.98	36 27 39.12	80	—13.64	+18.84
17.291647	Vienna	172 52 6.45	36 27 32.04	76	—16.05	+11.53
17.306382	Copenhagen	172 53 6.85	36 27 38.35	78–80	—20.55	+17.43
17.329512	Christiania	172 55 11.24	36 27 43.33		+ 1.84	+21.81
17.359841	Cambridge, Eng.	172 57 24.53	36 27 35.67	79	+ 1.18	+12.48
17.412500	Pulkova	173 1 3.30	36 27 37.43		—13.25	+14.35
17.532051	Ann Arbor	173 9 55.97	36 27 41.20	80	—13.00	+17.41
17.550766	"	173 11 18.70	36 27 39.60	78	—13.94	+15.88
18.287944	Königsberg	174 7 41.96	36 26 52.95	82	— 5.55	+10.13
18.317231	Copenhagen	174 9 47.28	36 26 36.12	82	—17.56	— 3.46
18.315490	Paris	174 9 50.67	36 27 5.47	82	—15.38	+25.92
18.319006	Copenhagen	174 10 16.79			+ 3.62	
18.319751	Liverpool	174 10 4.71	36 26 54.20	84	—11.95	+14.91
18.328100	"	174 10 44.18	36 26 50.64	84	—11.69	+12.29
18.336446	"	174 11 25.44	36 26 46.37	84	— 9.63	+ 8.98
18.351046	Markree	174 12 34.18	36 26 29.05	82	— 9.53	— 6.62
18.413297	Pulkova	174 17 27.00	36 26 37.87		—10.07	+ 9.83
18.424440	Christiania	174 18 19.92	36 26 48.38		— 9.80	+21.78
18.534230	Ann Arbor	174 26 57.07	36 26 28.80	80	—13.13	+17.32
18.545534	Christiania	174 27 44.11	36 26 24.43	82	—19.88	+14.61
19.272473	Kremsmünster	175 26 52.76	36 23 57.00	82	—13.13	+17.81
19.286802	Christiania	175 28 3.28	36 23 47.34	83-4	—14.10	+12.02
19.289696	Geneva	175 28 16.40	36 23 47.67	86	—15.42	+13.14
19.289697	Florence	175 28 16.03	36 23 46.81	86	—17.29	+12.36
19.312751	Paris	175 30 10.86	36 23 35.74	86	—18.14	+ 7.61
19.387398	Armagh	175 36 59.67	36 23 13.49	82	+18.68	+ 6.36
19.515182	Washington	175 47 20.76	36 22 40.93	82	— 4.56	+12.25
19.531160	Ann Arbor	175 48 29.01	+36 22 36.21	86	—17.26	+12.55

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Sept. 20.252856	Florence	176 51 5.56	+36 17 55.79	86	— 4.42	+ 9.65
20.265575	Kremsmünster	176 52 6.63	36 17 54.15	85–6	—10.89	+13.83
20.288793	Christiania	176 54 6.19	36 17 44.26	84	—14.68	+14.67
20.310506	Paris	176 55 56.55	36 17 32.58	86	—20.02	+13.09
20.344173	Durham	176 59 4.23		86	—11.91	
20.360918	Markree	177 1 4.66	36 17 10.97	85–6	+19.05	+15.35
20.362970	Durham		36 17 8.06	86		+13.43
20.470112	Christiania	177 10 18.96	36 16 12.75	84-7	—12.33	+10.69
20.510140	Washington	177 14 0.80	36 15 52.79	85	— 6.24	+11.01
20.510962	Cambridge, U. S.	177 13 56.28	36 15 55.65	86	—15.20	+14.28
20.514399	Ann Arbor	177 14 13.81	36 15 53.71	86	—16.22	+14.10
20.515778	Washington	177 14 31.25	36 15 53.84	86	— 6.23	+18.42
20.337975	Göttingen	176 58 20.29	36 17 15.66	86	—17.46	+ 8.64
20.647851	Durham	177 26 20.58		86	—12.96	
20.650006	Berlin	177 26 35.43	36 14 41.65	86	— 9.98	+13.48
20.663728	Durham		36 14 36.18	86		+15.39
20.680067	"	177 29 15.12		86	—14.02	
20.690908	Armagh	177 30 10.59	36 14 16.78	85	—17.72	+10.86
21.299000	Geneva	178 26 46.31	36 8 7.33	86	— 8.18	+18.35
21.299037	Bonn	178 26 39.87	36 8 3.23	88	—14.82	+14.28
21.311531	Königsberg	178 27 56.29	36 7 51.27	88	— 9.39	+11.08
21.303351	Liverpool	178 27 8.29	36 8 1.77	92	—10.92	+15.80
21.317379	Göttingen	178 28 8.88	36 7 55.28	87	—30.04	+19.07
21.325561	Christiania	178 29 7.94	36 7 51.07	87	—17.50	+20.55
21.326150	Göttingen	178 29 1.55		85	—27.25	
21.319350	Liverpool	178 28 41.22	36 7 52.96	92	— 8.91	+18.11
21.326150	Göttingen	178 28 56.49		86	—32.31	
21.328914	Leyden	178 29 34.02	36 7 45.85	89	—10.51	+17.68
21.334417	Copenhagen	178 30 4.85	36 8 1.66	88	— 8.99	+37.35
21.335339	Liverpool	178 30 13.96	36 7 44.14	92	— 7.24	+20.48
21.335411	Berlin	178 30 9.55	36 7 38.80	85	—11.95	+15.17
21.337062	Copenhagen	178 30 18.26	36 7 43.82		—12.65	+21.26
21.350418	Armagh	178 31 34.04	36 7 20.35	86	—12.93	+ 7.27
21.351112	Markree	178 31 35.38	36 7 20.64	85–6	—15.54	+ 8.05
21.364569	Cambridge, Eng.	178 33 34.25	36 7 18.18	85	+26.73	+15.09
21.375570	Leyden	178 33 51.15	36 7 10.58	86	—19.23	+15.29
21.471664	Christiania	178 43 5.72	36 5 55.02	86	—14.71	+ 9.06
22.287329	Kremsmünster	180 3 16.71	35 54 35.24	90	—12.99	+15.55
22.290700	Göttingen	180 3 28.68	35 54 35.97	90	—21.42	+19.49
22.294876	Bonn	180 3 56.12	35 54 25.39	90	—19.26	+12.90
22.305836	Göttingen	180 5 8.42	35 54 13.96	89	—13.35	+11.95
22.357669	Markree	180 10 12.18	35 53 14.67	90	—24.22	+ 2.69
23.266456	Kremsmünster	181 45 16.71	35 36 36.83	93	—10.04	+13.25
23.296747	Copenhagen	181 48 34.97	35 35 45.14	93	— 7.21	— 0.33
23.302829	Vienna	181 49 2.03		93	—19.44	
24.270615	"	183 36 39.67	35 12 43.13	94	—11.82	+12.79
24.279692	Liverpool	183 37 45.87	35 12 32.54	92	— 8.00	+16.83
24.290996	Königsberg	183 39 0.60	35 12 10.37	93	—12.01	+12.92
24.290137	Liverpool	183 38 55.72	35 12 14.66	92	— 9.98	+15.83
24.300586	"	183 40 7.05	35 11 57.07	92	—10.56	+15.16
24.332837	Greenwich	183 43 43.65	35 11 6.04		—16.20	+16.63
24.341926	Cambridge, Eng.	183 44 49.63	+35 10 44.54	93	—12.93	+10.00

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Sept. 24.422078	Christiania	183 53 54.89	+35 8 45.68	92–3	—22.26	+23.65
24.423542	Pulkova	183 54 10.95	35 8 29.39		—16.36	+ 9.80
25.261189	Kremsmünster	185 33 31.30	34 42 52.75	95	—13.68	+14.55
25.272704	Göttingen	185 35 5.21	34 42 29.31	97	—13.14	+14.41
25.275727	"	185 35 19.06	34 42 27.18	96	—21.39	+18.36
25.292244	Königsberg	185 37 26.04	34 41 49.07	95	—15.31	+13.75
25.285921	Liverpool	185 36 40.26	34 42 2.64	98	—14.79	+14.49
25.292886	"	185 37 27.31	34 41 49.79	98	—18.74	+15.78
25.299857	"	185 38 20.05	34 41 35.14	98	—17.07	+15.30
25.314011	Berlin	185 40 15.72	34 41 8.62	97	— 5.15	+17.63
25.325785	Cambridge, Eng.	185 41 23.38	34 40 48.39	95	—23.87	+21.46
25.352642	Greenwich	185 44 52.20	34 39 47.44		—12.29	+15.24
25.489007	Cambridge, U. S.	186 1 44.71	34 35 5.39	95	— 6.54	+18.45
25.520255	Ann Arbor	186 5 36.19	34 33 59.05	96	— 6.93	+18.58
25.648289	Durham	186 21 26.18	34 29 18.03	95	—11.63	+14.59
25.667479	"	186 23 47.69	34 28 37.93	95	—13.85	+16.65
26.298863	Geneva	187 44 6.99	34 3 46.23	98	—18.40	+11.15
26.306382	Bonn	187 45 11.32	34 3 29.86	99	—12.62	+13.69
26.484109	Christiania	188 8 21.71	33 55 50.83	98	—13.95	+ 9.52
26.527307	Washington	188 14 4.57	33 53 58.36	99	—11.54	+10.03
26.531339	Ann Arbor	188 14 32.82	33 53 52.01	99	—15.15	+14.33
26.322762	Durham	187 47 18.92		98	—12.67	
26.339268	"	187 49 25.98		98	—14.35	
26.346136	"		34 1 52.60	98		+16.76
27.242312	Kremsmünster	189 50 3.90	33 20 26.11	100	—15.03	+15.02
27.278053	Florence	189 54 52.02	33 18 29.13	100	—21.23	+ 6.36
27.279344	Bonn	189 55 13.95	33 18 31.70	100	— 9.94	+12.86
27.286224	Geneva	189 56 5.07	33 18 6.62	100	—15.56	+10.26
27.293169	Liverpool	189 56 45.38	33 17 50.51	103	—32.62	+13.76
27.320928	Cambridge, Eng.	190 0 55.60	33 16 25.90	100	—11.51	+14.06
27.360644	Christiania	190 6 22.42	33 14 25.88	100	—13.25	+16.27
27.368878	Armagh	190 8 5.00	33 12 51.72	100	+21.11	+ 7.56
28.259639	Kremsmünster	192 13 38.59	32 24 1.27	105	—14.00	+13.79
28.279187	Bonn	192 16 33.12	32 22 51.95	102	— 9.72	+15.84
28.292041	Paris	192 18 24.52	32 22 7.66	105	—10.37	+18.61
28.299010	Copenhagen	192 19 25.03	32 22 47.15	102	—10.64	+83.66
28.307004	"	192 20 17.27			—28.15	
28.326205	Durham	192 23 30.97	32 20 2.45	101	— 2.10	+19.01
28.506643	Washington	192 49 51.07	32 9 57.40	105	— 7.10	+90.12
29.260515	Kremsmünster	194 42 44.06	31 17 38.66	104	—15.74	+17.75
29.263858	Vienna	194 43 20.04	31 17 22.20	112–3	—10.41	+15.78
29.268433	Königsberg	194 43 55.92	31 17 3.25	109	—16.49	+16.68
29.281328	Leyden	194 46 4.00		107	— 6.72	
29.284939	Florence	194 46 20.95	31 15 58.15	106	—22.91	+23.32
29.293825	Göttingen	194 47 12.72	31 14 27.20	108	—52.74	+31.07
29.303599	Vienna	194 49 25.47	31 14 26.76	104	— 9.77	+13.28
29.323978	Geneva	194 52 22.65	31 12 49.99	104	—10.75	+ 1.24
29.498277	Cambridge, U. S.	195 19 17.55	31 0 5.74	112	—15.54	+16.24
29.505309	Washington	195 20 39.97	30 59 33.17	112	+ 1.62	+15.39
29.708542	Greenwich	195 52 4.65	30 43 56.43		— 9.47	+11.88
30.207585	Pulkova	197 10 52.77	30 3 30.84	115	—15.05	+12.96
30.227222	"	197 14 4.05	+30 1 54.13	115	—12.25	+15.75

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δ α	Δ δ
		° ′ ″	° ′ ″		″	″
Sept. 30.257998	Kremsmünster	197 18 58.29	+29 59 16.75	114	—13.70	+14.94
30.263155	Liverpool	197 19 53.58	29 58 48.95	111	— 8.03	+13.44
30.270043	Geneva	197 20 44.43	29 58 7.41	114	—14.94	+ 7.07
30.273627	Liverpool	197 21 33.97	29 57 55.89	111	— 8.37	+13.88
30.284093	"	197 23 11.36	29 57 0.83	111	—11.71	+12.36
30.295506	Geneva	197 24 59.97	29 56 4.30	110	—13.00	+14.33
30.314046	Markree	197 27 57.13	29 54 40.09	116	—14.49	+25.37
30.320418	Armagh	197 28 27.61	29 53 52.46	114	—45.43	+10.53
30.335823	Cambridge, Eng.	197 31 27.60	29 52 40.33	116	—14.03	+17.84
30.342848	Armagh	197 32 10.53	29 52 7.19	116	—38.89	+20.97
30.501681	Washington	197 58 19.42	29 38 11.42	116	— 8.92	+16.37
30.533553	"	198 4 1.45	29 35 4.29	117	+22.99	— 1.49
30.323109	Durham	197 29 30.83		116	— 8.17	
30.334899	"		29 52 43.32	116		+16.06
Oct. 1.268006	Kremsmünster	200 4 28.15	28 26 19.30	122	—13.99	+10.99
1.295157	Greenwich	200 9 6.30	28 23 40.45		— 8.55	+13.98
1.300854	Göttingen	200 9 54.20	28 23 4.11	122	—17.99	+11.72
1.300854	"	200 9 56.05	28 23 12.04	123	—16.14	+19.65
1.311247	Geneva	200 11 43.22	28 22 0.09	122	—13.36	+ 8.83
1.312189	Leyden	200 11 57.08		124	— 8.79	
1.350841	Christiania	200 18 34.24	28 18 7.24	121	— 2.79	+16.49
1.507696	Washington	200 44 39.17	28 2 23.24	120	—20.96	+26.08
2.263880	Königsberg	202 54 21.88	26 40 19.60	125	— 9.63	+18.78
2.264704	Vienna	202 54 35.63	26 40 0.05	126	— 8.62	+ 7.72
2.268121	Kremsmünster	202 55 2.46	26 39 43.59	126	—14.25	+12.60
2.285332	Greenwich	202 58 7.20	26 37 48.19		— 8.83	+15.30
2.286363	Geneva	202 58 16.03	26 37 35.34	128	—10.74	+ 9.53
2.287888	Leyden	202 58 36.73		126	— 5.95	
2.300320	Christiania	203 0 39.16	26 36 4.47	125	—13.12	+14.64
2.300320	"	203 0 42.22	26 36 3.14	126	—10.06	+13.31
2.304222	Geneva	203 1 16.27	26 35 31.18	125	—16.69	+10.39
2.304203	Florence	203 1 24.76	26 35 33.99	125	— 9.04	+11.60
2.509941	Washington	203 37 20.32	26 11 41.67	127	—13.33	+20.39
3.257362	Christiania	205 49 35.43	24 39 9.16	118	—11.51	+21.07
3.270639	Kremsmünster	205 51 58.36	24 37 15.00	132	—11.16	+10.48
3.272642	Vienna	205 52 16.67	24 36 58.69	130	—14.36	+ 9.83
3.284526	Bonn	205 54 28.85	24 35 30.35	129	— 9.85	+14.36
3.290934	Geneva	205 55 35.48	24 34 31.04	133	—12.08	+ 5.19
3.308712	"	205 58 49.85	24 32 15.86	130	— 8.82	+ 9.33
3.530868	Washington	206 38 46.64	24 2 50.88	131	— 7.97	+11.12
4.224497	Berlin	208 44 54.47	22 25 47.75	134	— 7.10	+13.51
4.243681	Vienna	208 48 26.32	22 23 13.59	134	— 6.14	+27.13
4.251518	Liverpool	208 49 55.98	22 21 51.55	138	— 2.64	+13.74
4.258040	Vienna	208 50 59.54	22 20 51.68	135	—10.81	+11.04
4.262837	Kremsmünster	208 51 55.80	22 20 11.16	134	— 7.31	+12.59
4.265508	Liverpool	208 52 31.20	22 19 47.54	138	— 1.30	+12.42
4.279468	"	208 55 1.54	22 17 46.22	138	— 4.51	+13.71
4.283145	Göttingen	208 55 45.53	22 17 23.01	136	— 0.99	+22.82
4.287557	Leyden	208 56 37.45	22 16 33.54	136	+ 2.39	+12.16
4.299512	Durham	208 58 50.63		136	+ 4.00	
4.303897	Geneva	208 59 18.72	22 14 4.14	134	—16.18	+ 6.66
4.308288	Durham		+22 13 32.19	136		+13.42

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Oct. 4.316911	Durham	209 2 0.96		136	+ 2.78	
4.334335	Markree	209 5 7.55	+22 9 33.66	136	— 2.52	+ 4.91
5.204358	Berlin	211 46 12.01	19 55 14.35	138	+ 1.71	+ 4.58
5.204722	Vienna	211 46 5.12	19 55 21.54	138	— 9.23	+15.30
5.205456	Pulkova	211 46 13.80	19 55 12.84	138	— 8.74	+13.68
5.217774	Königsberg	211 48 32.30	19 53 12.25	138	— 7.91	+12.39
5.218247	Pulkova	211 48 39.50	19 53 7.27	138	— 6.00	+12.01
5.219959	Vienna	211 48 59.87	19 52 49.10	138	— 4.78	+10.46
5.234935	Göttingen	211 51 42.14	19 50 27.26	138	— 9.90	+13.85
5.249079	Breslau	211 54 28.07	19 47 58.98	138	— 2.12	+ 2.97
5.253226	Geneva	211 55 4.97	19 47 24.09	138	—11.59	+13.40
5.254982	Christiania	211 55 26.91	19 47 14.03	138	— 9.27	+15.42
5.275980	Bonn	211 59 23.37	19 43 47.78	138	— 7.65	+13.59
5.285601	Greenwich	212 1 9.15	19 42 12.83		— 9.49	+12.44
5.285703	Markree	212 1 17.04	19 42 15.95	138	— 2.75	+16.56
5.291148	Armagh	212 2 17.13	19 41 13.71	138	— 3.60	+ 7.46
5.299001	Geneva	212 3 38.23	19 40 2.00	139	—10.33	+12.40
5.302480	Cambridge, Eng.	212 4 21.40	19 39 23.24	138	— 6.07	+ 7.63
5.313001	Durham	212 6 20.02		139	— 5.20	
5.324932	"		19 35 48.01	139		+12.00
5.506095	Washington	212 42 19.31	19 5 58.72	140	—10.13	+11.89
5.513707	Ann Arbor	212 43 46.90	19 4 42.00	137	— 7.94	+11.00
6.223085	Berlin	214 56 56.76	17 3 5.99	143	— 4.30	+12.95
6.243362	Christiania	215 0 43.33	16 59 31.04	143	— 6.57	+13.15
6.245121	Breslau	215 1 0.08	16 59 5.37	143	— 9.68	+ 6.18
6.259660	Göttingen	215 3 45.60	16 56 40.65	143	— 8.24	+15.76
6.268458	Geneva	215 5 21.56	16 55 2.18	142	—11.60	+11.08
6.274580	"	215 6 31.88	16 53 58.80	141	—10.39	+12.86
6.276532	Göttingen	215 6 52.90		144	—11.40	
6.281182	Copenhagen	215 8 6.09	16 52 47.79		+ 9.30	+12.14
6.283378	Leyden	215 8 13.91		143	— 7.67	
6.292062	Kremsmünster	215 9 48.62	16 50 49.68	144	—11.01	+10.00
6.295746	Paris	215 10 30.22	16 50 16.69	142	—10.98	+16.28
7.223698	Berlin	218 5 29.88	13 59 20.99	148	— 4.97	+ 8.83
7.233783	Vienna	218 7 27.10	13 57 27.63	149	— 1.74	+10.48
7.243249	Kremsmünster	218 9 8.42	13 55 40.48	148	— 7.50	+11.42
7.245732	Vienna	218 9 37.35	13 55 12.90	149	— 6.61	+12.24
7.246070	Breslau	218 9 31.48	13 55 2.98	147–9	—16.35	+ 6.16
7.262217	Kremsmünster	218 12 44.56	13 52 3.43	148	— 5.82	+10.75
7.283573	Geneva	218 16 46.97	13 47 59.50	146	— 4.87	+11.53
7.285147	"	218 16 58.14	13 47 42.69	151	—11.50	+12.74
7.291893	Göttingen	218 18 16.98	13 46 28.98	148	— 8.93	+16.28
7.301455	Florence	218 20 4.61	13 44 24.33	146	— 9.40	+ 1.19
7.307955	Durham	218 21 19.72	13 43 39.05	146	— 7.77	+30.43
7.319152	Markree	218 23 26.57	13 41 6.23	151	— 7.51	+ 6.05
7.326498	Armagh	218 24 50.37	13 39 49.65	151	— 6.74	+13.78
8.228454	Breslau	221 14 27.66	10 42 54.76	152–3	— 6.84	+25.18
8.231321	Kremsmünster	221 15 4.35	10 42 8.57	152	— 2.40	+13.61
8.243545	Bonn	221 17 19.32	10 39 36.69	154	— 4.85	+ 9.15
8.264194	Göttingen	221 21 14.02		155	— 2.27	
8.266683	Paris	221 21 41.47	10 35 3.07	152	— 2.80	+14.90
8.266998	Liverpool	221 21 35.40	+10 34 51.86	150	—12.42	+ 7.52

6

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	$\Delta\alpha$	$\Delta\delta$
		° ′ ″	° ′ ″		″	″
Oct. 8.275565	Göttingen		+10 33 7.11	155		+ 6.28
8.276015	Florence	221 23 17.71	10 32 56.76	152	—11.46	+ 1.37
8.279737	Altona	221 23 58.29	10 32 12.74	154	—12.70	+ 2.31
8.279866	Leyden	221 24 12.91		152	+ 0.49	
8.294975	Liverpool	221 26 57.68	10 29 16.75	150	— 4.56	+10.59
8.296998	Armagh	221 27 26.35	10 28 51.13	152	+ 1.39	+ 9.44
8.298771	Cambridge, Eng.	221 27 34.94	10 28 30.68	154	— 9.95	+ 9.42
8.311355	Leyden		10 25 59.79	152		+11.83
8.347951	Markree	221 36 20.06	10 18 31.69	154	—37.30	+ 7.08
8.994392	Batavia	223 37 18.09	8 6 22.90		—14.23	+19.45
9.231500	Christiania	224 21 25.29	7 16 55.62	174	— 4.68	+10.06
9.233137	Pulkova	224 21 41.92	7 16 33.77	158	— 6.23	+ 8.73
9.253497	Göttingen	224 25 39.65	7 12 15.59	159	+ 5.57	+ 5.52
9.263202	Altona	224 27 9.79	7 10 16.79	159–62	—11.95	+ 8.30
9.266902	Königsberg	224 27 50.13	7 9 31.58	159	—12.66	+ 9.47
9.268608	Greenwich	224 28 17.10	7 9 10.90		— 0.68	+10.16
9.275645	Geneva	224 29 38.63	7 7 40.58	165	— 1.14	+ 8.04
9.277736	Göttingen	224 30 7.98	7 7 15.21	159	+ 5.03	+ 8.87
9.282223	Paris	224 30 48.43	7 6 17.56	156	— 4.29	+ 7.49
9.282566	Geneva	224 30 43.79	7 6 7.43	164	—12.73	+ 1.66
9.312163	Cambridge, Eng.	224 36 20.97	6 59 59.69	161	— 3.69	+ 5.21
9.495923	Washington	225 10 50.23	6 20 51.92	157	+31.34	—31.71
9.496575	Ann Arbor	225 10 26.00	6 21 20.16	160	— 0.09	+ 4.63
10.224865	Christiania	227 23 39.55	3 47 17.21	176	—20.35	+ 3.77
10.254573	Kremsmünster	227 28 53.28	3 40 59.55	166	— 6.49	+ 5.32
10.267580	Altona	227 31 12.00	3 38 15.43	166	— 9.19	+ 7.15
10.309878	Armagh	227 38 46.19	3 29 15.99	163	—14.63	+ 7.59
10.987956	Batavia	229 40 28.31	1 4 11.49		—16.69	—28.46
11.238792	Kremsmünster	230 25 8.58	0 11 16.82	169	— 4.78	+ 1.73
11.266800	Vienna	230 29 55.06	0 5 27.48	168	—14.97	+ 9.92
11.274023	Cape 1	230 31 11.64	0 3 52.07	168	—14.85	+ 6.91
11.274066	Greenwich	230 31 24.90	0 3 49.50		— 2.05	+ 3.69
11.296548	Armagh	230 35 27.26	+0 0 6.50	167	+ 2.42	+50.76
11.297047	Leyden	230 35 7.51	—0 1 3.06	168	—22.62	+ 5.38
11.310513	Markree	230 37 48.17	0 3 57.06	168	— 4.36	+ 3.21
11.313423	Cambridge, Eng.	230 38 18.92	0 4 37.88	170	— 4.38	— 0.49
12.269342	Altona	233 24 24.30	3 26 22.83	171-3	+ 3.31	+ 4.99
12.270434	Cape 1	233 24 0.80	3 26 34.39	172	—31.39	+ 6.08
12.280254	Paris	233 26 8.65	3 28 33.88	175	— 4.17	+10.85
12.982945	Batavia	235 24 35.70	5 54 11.79		— 6.64	— 0.68
13.213628	Pulkova	236 2 43.19	6 40 46.99	181	—12.35	+24.10
13.255055	Geneva	236 9 39.78	6 49 28.88	180	— 5.28	+ 5.65
13.277955	Liverpool	236 13 34.28	6 54 6.25	182	+ 3.14	+ 6.41
13.287219	Florence	236 14 0.26	6 54 37.67	180	+ 6.15	+ 3.25
13.288445	Liverpool	236 15 19.72	6 56 14.72	182	+ 5.08	+ 5.27
13.488816	Washington	236 48 8.43	7 36 41.60	177	+ 4.97	— 0.12
13.508620	Ann Arbor	236 51 16.00	7 40 41.80	179	— 1.20	— 2.08
14.234268	Geneva	238 47 40.71	10 3 39.86	186	— 4.92	+ 0.52
14.256363	"	238 51 12.38	10 8 0.59	190	— 2.61	— 1.21
14.256363	"	238 51 9.36	10 7 56.95	194	— 5.63	— 4.85
14.261041	Kremsmünster	238 51 55.57	10 8 50.99	194	— 3.73	+ 0.04
14.271791	Cape 1	238 53 35.95	—10 10 50.70	183	— 5.11	+ 4.95

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Oct. 14.287875	Cape 1	238 56 7.05	—10 13 58.57	183	— 6.17	+ 3.38
14.528756	Ann Arbor	239 34 4.89	11 0 10.88	184	+ 5.45	+ 2.62
15.239372	Geneva	241 22 54.89	13 12 46.38	197	— 6.90	+ 4.10
15.240280	Vienna	241 23 7.16	13 12 59.52	188	— 2.84	+ 0.88
15.241461	Göttingen	241 23 24.24	13 13 8.86	191	+ 3.55	+ 4.49
15.243991	Kremsmünster	241 23 37.26	13 13 39.29	197	— 6.32	+ 0.75
15.247077	Göttingen	241 24 14.36	13 14 21.93	189	+ 2.84	— 7.09
15.256351	Florence	241 25 31.52	13 15 55.77	196	— 3.90	+ 0.56
15.256407	Liverpool	241 25 34.18	13 15 57.40	213	— 1.77	— 0.45
15.259106	Geneva	241 25 59.61	13 16 21.25	185	— 0.75	+ 5.25
15.286250	Cambridge, Eng.	241 29 56.64	13 21 30.43	196	— 9.08	— 7.24
15.316533	Cape 1	241 34 25.22	13 26 51.51	193	—13.86	+ 2.14
15.499105	Washington	242 1 57.44	13 59 51.88	195	— 1.47	+20.70
15.503998	Ann Arbor	242 2 46.28	14 0 45.35	187	+ 2.56	— 0.05
16.231845	Breslau	243 49 15.80	16 9 2.62	200	+ 0.19	—47.67
16.234474	Bonn	243 49 35.41	16 8 44.87	199	— 2.88	— 2.97
16.234960	Berlin	243 49 43.56	16 8 48.85	198	+ 1.21	— 1.96
16.240516	Vienna	243 50 31.08	16 9 49.33	200	+ 0.70	— 5.59
16.242347	Göttingen	243 50 43.61	16 10 9.11	199	— 2.56	— 6.59
16.245888	Altona	243 51 20.48	16 10 28.13	199	+ 3.80	+10.62
16.257474	Florence	243 53 1.93	16 12 51.81	201	+ 5.43	—14.54
16.270196	Cambridge, Eng.	243 54 41.20	16 14 42.87	201	— 4.86	+ 4.44
16.271260	Paris	243 54 55.08	16 14 55.92	201	— 0.15	+ 2.24
16.277275	Cape 1	243 55 45.29	16 15 58.81	192	— 1.70	+ 0.82
16.287908	Armagh	243 57 24.89	16 16 35.73	201	+ 6.42	+72.43
16.288004	Cape 2	243 57 13.13	16 17 47.31	201	— 6.17	+ 1.83
16.292798	Markree	243 58 0.03	16 18 41.71	201	— 0.48	— 3.67
16.296702	Cape 1	243 58 31.73	16 19 17.80	192	— 2.38	+ 0.08
16.318776	Cape 2	244 1 33.08	16 23 0.71	201	—10.66	+ 2.06
16.501781	Washington	244 27 47.24	16 53 51.26	199	— 0.90	+ 2.28
17.224979	Vienna	246 8 24.59	18 51 37.14	203	— 0.96	+ 2.68
17.237201	"	246 10 5.33	18 53 39.33	203	— 3.73	+ 0.43
17.258608	Geneva	246 12 59.23	18 57 1.17	206	— 5.03	+ 1.73
18.000873	Batavia	247 52 5.86	20 50 46.13		—14.59	+ 9.25
18.229243	Vienna	248 21 57.50	21 24 30.24	205	— 8.58	+ 5.21
18.241802	Geneva	248 23 37.72	21 26 26.89	202	— 5.93	— 0.43
18.259592	"	248 25 58.24	21 28 54.26	209	— 3.49	+ 6.80
18.270142	Cape 1	248 27 23.57	21 30 31.30	204	+ 0.03	+ 1.92
18.285490	Cape 2	248 29 21.63	21 32 45.94	207	— 0.86	+ 1.21
18.285490	"	248 29 21.93	21 32 42.07	209	— 0.56	+ 5.08
18.285797	Cape 1	248 29 26.42	21 32 44.76	204	+ 1.54	+ 1.35
18.299314	Cape 2	248 31 6.06	21 34 45.36	207	— 3.49	+ 2.28
18.299314	"	248 31 7.26	21 34 42.86	209	— 2.29	+ 4.78
18.305006	Cape 1	248 31 53.66	21 35 35.47	204	+ 0.05	+ 1.74
18.322705	Cape 2	248 33 53.28	21 38 9.85	207	—17.23	+ 1.34
18.322705	"	248 33 54.33	21 38 7.77	209	—16.18	+ 3.42
19.002366	Batavia	250 0 7.86	23 13 49.14		— 1.78	+ 3.58
19.452976	Cambridge, U. S.	250 55 27.49	24 14 13.54	210	+ 1.62	+ 4.77
19.494892	Washington	251 0 32.11	24 19 46.55	208	+ 1.94	+ 1.80
20.454666	Cambridge, U. S.	252 53 30.05	26 20 15.44	211	+ 1.39	+ 5.44
21.264748	Cape 2	254 24 12.23	27 54 18.61	212	— 3.14	+ 2.07
21.264748	"	254 24 15.98	—27 54 18.64	214	+ 0.61	+ 2.04

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Oct. 21.275041	Cape 1	254 25 20.30	—27 55 28.28	216	— 2.70	+ 1.42
21.278415	Cape 2	254 25 43.22	27 55 49.45	212	— 1.95	+ 2.88
21.278415	"	254 25 39.32	27 55 53.33	214	— 5.85	— 1.00
21.291348	"	254 27 0.37	27 57 19.44	212	— 9.70	— 0.48
21.291348	"	254 27 5.32	27 57 24.08	214	— 4.75	— 5.12
21.291348	"	254 27 2.92	27 57 20.73	218	— 7.15	— 1.77
21.304881	Cape 1	254 28 34.31	27 58 49.18	216	— 4.56	+ 0.33
21.305007	Cape 2	254 28 34.76	27 58 46.55	212	— 4.94	+ 3.80
21.305007	"	254 28 25.41	27 58 50.29	214	—14.29	+ 0.06
21.312209	Cape 1	254 29 23.87	27 59 37.80	217	— 3.06	+ 0.69
21.331513	"	254 31 31.57	28 1 45.93	217	— 1.83	+ 1.44
21.500843	Ann Arbor	254 49 54.78	28 20 33.27	215	— 2.26	— 4.99
22.239002	Florence	256 7 51.54	29 39 14.04	223	—17.10	—35.31
22.278702	Cape 2	256 12 11.95	29 42 41.71	221	— 3.20	+ 0.20
22.278702	"	256 12 10.36	29 42 40.11	222	— 4.79	+ 1.80
22.278702	"	256 12 11.41	29 42 41.16	223	— 3.74	+ 0.75
22.284269	Cape 1	256 12 49.03	29 43 16.82	220	— 1.12	— 0.42
22.289345	Cape 2	256 13 23.57	29 43 44.51	221	+ 1.90	+ 2.94
22.289345	"	256 13 20.18	29 43 45.91	222	— 1.49	+ 1.54
22.289345	"	256 13 23.03	29 43 43.77	223	+ 1.36	+ 3.68
22.290592	Cape 1	256 13 29.05	29 43 52.60	219	— 0.36	+ 0.65
22.299591	Cape 2	256 14 22.83	29 44 47.54	221	— 2.45	+ 2.54
22.299591	"	256 14 21.09	29 44 48.28	222	— 4.19	+ 1.80
22.299591	"	256 14 18.69	29 44 48.80	223	— 6.59	+ 1.28
22.309611	"	256 15 25.94	29 45 48.96	221	— 1.50	+ 2.34
22.309611	"	256 15 25.70	29 45 49.95	222	— 1.74	+ 1.35
22.309611	"	256 15 29.45	29 45 50.73	223	+ 2.01	+ 0.57
22.312775	Cape 1	256 15 45.68	29 46 10.47	219	— 1.38	+ 0.14
22.318450	"	256 16 18.42	29 46 46.33	220	— 3.82	— 1.08
22.319024	Cape 2	256 16 17.61	29 46 45.72	221	— 8.19	+ 3.03
22.319024	"	256 16 10.47	29 46 47.92	222	—15.33	+ 0.83
22.319024	"	256 16 1.92	29 46 42.31	223	—23.88	+ 6.44
22.483971	Ann Arbor	256 33 22.80	30 3 31.72	221	+ 0.15	— 5.17
23.267205	Cape 2	257 51 40.72	31 19 14.31	224	—11.29	— 2.77
23.267205	"	257 51 53.62		226	+ 1.61	
23.280606	"	257 52 53.95	31 20 26.12	224	—16.88	+ 0.29
23.280606	"	257 52 43.45	31 20 26.39	226	— 6.38	+ 0.02
23.288695	"	257 53 48.86	31 21 9.07	224	— 9.51	+ 2.67
23.288695	"	257 53 57.11	31 21 12.89	226	— 1.26	— 0.15
23.289249	Cape 1	257 53 56.37	31 21 13.60	225	— 4.25	+ 1.24
23.295526	Cape 2	257 54 21.53	31 21 52.66	224	—16.96	— 2.76
23.295526	"	257 54 30.23	31 21 50.95	226	— 8.26	— 1.05
23.302760	"	257 55 8.13	31 22 29.71	224	—12.86	+ 0.57
23.302760	"	257 55 26.13	31 22 28.43	226	+ 5.14	+ 1.85
23.309794	Cape 1	257 55 58.76	31 23 10.07	225	— 3.52	— 0.54
23.327418	"	257 57 41.92	31 24 48.95	225	— 3.74	— 1.20
24.270223	"	259 27 32.31	32 48 36.57	227	— 1.42	+ 5.01
24.291839	"	259 29 32.03	32 50 31.24	227	— 2.03	+ 0.85
24.297234	Cape 2	259 29 49.22	32 51 1.49	227	—14.85	— 1.86
24.310027	"	259 31 11.13	32 52 4.42	227	— 4.07	+ 0.48
24.310027	"	259 31 16.53	32 52 6.29	228	+ 1.33	— 1.39
24.310359	Cape 1	259 31 13.53	—32 52 5.50	227	— 3.51	+ 1.09

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Oct. 24.320564	Cape 2	259 31 42.71	—32 53 5.54	227	—31.04	— 6.94
24.320564	"	259 31 52.61	32 53 1.05	228	—21.14	— 2.45
24.330842	"	259 32 59.55	32 54 0.14	227	—11.29	— 9.20
24.330842	"	259 33 9.30	32 53 48.45	228	— 1.54	+ 2.49
25.227393	Florence	260 53 53.91	34 6 59.10	229	—15.89	— 2.01
25.270231	Cape 1	260 57 54.27	34 10 16.66	229	— 1.94	+ 1.42
25.271213	Cape 2	260 57 57.74	34 10 19.08	229	— 3.65	+ 3.58
25.282375	"	260 58 58.75	34 11 12.24	229	— 6.54	+ 2.67
25.291788	Cape 1	260 59 46.79	34 11 57.90	229	— 3.15	+ 1.03
25.293562	Cape 2	260 59 59.88	34 12 3.38	229	+ 0.58	+ 3.85
25.304389	"	261 0 50.73	34 12 54.18	229	— 5.64	+ 3.63
25.315282	"	261 1 46.80	34 13 45.56	229	— 6.95	+ 3.09
25.318415	Cape 1	261 2 5.27	34 14 1.35	230	— 4.99	+ 1.91
26.278599	Cape 2	262 24 18.84	35 25 45.54	232	— 1.14	— 7.08
26.284208	Cape 1	262 24 45.07	35 26 1.73	231	— 3.00	+ 0.82
26.293092	Cape 2	262 25 33.24	35 26 38.90	231	+ 0.71	+ 1.72
26.302328	Cape 1	262 26 16.43	35 27 19.84	231	— 2.28	+ 0.33
26.314169	Cape 2	262 27 11.32	35 28 10.74	231	— 6.59	+ 0.09
26.326702	"	262 28 10.19	35 28 58.26	231	— 5.34	+ 1.76
27.280555	Cape 1	263 45 43.72		234	— 5.26	
27.284622	Cape 2	263 45 51.16	36 34 26.62	233	—17.17	+ 6.40
27.290788	Cape 1	263 46 31.03	36 34 55.06	234	— 6.50	+ 2.22
27.302058	Cape 2	263 47 16.61	36 35 37.46	233	—14.68	+ 4.15
27.309387	Cape 1	263 47 59.50	36 36 7.06	234	— 6.51	+ 3.35
27.317984	Cape 2	263 48 44.14	36 36 36.09	233	— 2.71	+ 8.08
28.281355	"	265 2 58.83	37 37 9.95	236	—12.43	+ 7.24
28.291188	Cape 2	265 3 51.38	37 37 48.73	236	— 4.37	+ 4.07
28.296537	Cape 1		37 38 9.03	236		+ 3.13
28.317088	"	265 5 48.72	37 39 21.58	235	— 4.09	+ 4.86
28.326112	"	265 6 30.33	37 39 57.60	236	— 3.24	+ 1.42
29.272940	"	266 16 5.94	38 34 46.18	237	— 5.26	+ 0.76
29.280063	Cape 2	266 16 37.16	38 35 8.57	237	— 4.75	+ 2.16
29.291846	Cape 1	266 17 28.09	38 35 49.21	237	— 4.61	+ 0.85
29.291909	Cape 2	266 17 24.19	38 35 49.05	237	— 8.79	+ 1.22
29.304695	"	266 18 16.89	38 36 32.27	237	—11.16	+ 0.63
29.307344	Cape 1	266 18 35.52		237	— 3.93	
29.331186	"	266 20 15.91	38 37 59.43	238	— 6.16	+ 1.65
30.301428	Cape 2	267 28 16.21	39 29 46.29	239	— 4.66	— 0.52
30.311740	Cape 1	267 28 57.38	39 30 14.23	240	— 5.90	+ 3.28
30.318273	Cape 2	267 29 22.79	39 30 30.26	240	— 7.29	+ 7.34
30.328224	Cape 1	267 30 5.90	39 31 6.00	240	— 5.04	+ 2.19
30.530435	Santiago	267 43 57.03	39 41 20.96	240	— 0.09	+ 3.60
31.295670	Cape 1	268 34 49.84	40 18 43.51	242	+ 0.67	+ 2.96
31.298280	Cape 2	268 34 53.81	40 18 50.34	241	—12.70	+ 8.71
31.306177	Cape 1	268 35 53.96	40 19 35.56	242	— 6.42	+ 2.56
31.319299	Cape 2	268 36 17.65	40 19 54.19	241	—11.41	+ 4.72
31.328081	Cape 1	268 36 56.93	40 20 20.59	242	— 6.60	+ 3.30
31.551379	Santiago	268 51 36.28	40 30 49.08	241	+ 0.85	+ 4.52
31.568757	"	268 52 32.10		241	—10.74	
Nov. 2.369050	Cape 2	270 44 42.82	41 50 6.22	243	— 7.41	+ 0.79
2.516770	Santiago	270 53 37.97		244	— 2.09	
3.277366	Cape 1	271 38 12.72	—42 25 55.35	247	— 6.45	+ 2.12

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δ α	Δ δ
		° ′ ″	° ′ ″		″	″
Nov. 3.278903	Cape 2	271 38 19.65	—42 25 58.20	248	— 4.84	+ 2.80
3.293061	"	271 39 11.22	42 26 30.51	248	— 2.38	+ 2.93
3.309289	Cape 1	271 40 0.63	42 27 7.70	246	— 9.23	+ 2.87
3.317423	"		42 27 24.68	246		+ 4.50
3.536300	Santiago	271 53 6.60	42 35 42.91	251	— 6.38	+ 2.80
3.624757	"	271 58 11.93	42 38 59.81	251	— 4.32	+ 4.40
3.630955	"	271 58 26.58		253	—10.33	
4.276278	Cape 2	272 34 43.01	43 2 49.28	252	—15.65	+ 0.53
4.288704	Cape 1	272 35 31.25	43 3 12.07	250	— 8.89	+ 4.31
4.298756	Cape 2	272 36 7.70	43 3 37.54	252	— 5.98	+ 0.32
4.303762	Cape 1	272 36 22.17	43 3 43.33	250	— 8.22	+ 5.22
4.535387	Santiago	272 49 6.15	43 11 55.03	245	—13.43	+ 4.29
5.284601	Cape 2	273 29 48.34	43 37 27.19	254	—14.84	+ 8.22
5.309511	"	273 31 5.05	43 38 16.31	254	—18.25	+ 8.87
5.542071	Santiago	273 43 38.20	43 46 3.74	249	— 9.54	+ 2.16
6.064230	Batavia	274 11 8.70	44 2 27.45	254	—14.63	+31.54
6.284801	Cape 2	274 22 42.76	44 9 49.79	256	— 2.12	+ 1.59
6.290045	Cape 1	274 22 50.80	44 9 57.28	255	—10.37	+ 3.92
6.304504	Cape 2	274 23 38.85	44 10 30.04	256	— 7.22	— 1.77
6.305375	Cape 1	274 23 37.36	44 10 25.97	255	—11.42	+ 3.93
6.532629	Santiago	274 35 19.30	44 17 27.91	254	—12.16	+ 4.14
7.280508	Cape 2	275 13 15.71	44 39 53.10	257	—10.70	+ 6.01
7.292731	"	275 13 57.76	44 40 14.11	257	— 2.37	+ 6.49
7.303347	"	275 14 19.35	44 40 33.40	257	—15.66	+ 5.83
7.309731	Cape 1	275 14 42.67	44 40 44.70	257	—11.50	+ 5.77
7.344515	"	275 16 28.11		257	—10.70	
9.323115	Cape 2	276 51 59.05	45 36 11.76	258	—29.63	+ 5.23
9.335328	Cape 1	276 52 32.98	45 36 31.33	258	—30.19	+ 4.69
9.335882	Cape 2	276 52 51.74	45 36 26.63	258	—12.93	+10.22
9.352831	Cape 1	276 53 38.95		258	—13.47	
9.518349	Santiago	277 1 19.81	45 41 16.09	258	—17.66	+ 2.74
9.543581	"	277 2 33.13	45 41 56.89	258	—15.04	+ 0.70
11.282643	Cape 1	278 21 46.97	46 24 14.60	259	—13.47	+ 3.24
11.297551	"	278 22 26.77	46 24 34.49	259	—13.42	+ 4.06
11.314855	"	278 23 12.24		259	—14.09	
11.538432	Santiago	278 33 5.98	46 30 10.01	261	—14.46	+ 0.97
12.285219	Cape 1	279 5 46.91	46 46 51.73	260	—12.96	+ 2.35
12.301348	Cape 2	279 6 37.03	46 47 11.97	262	— 4.75	+ 3.34
12.304447	Cape 1	279 6 35.60	46 47 16.71	260	—14.23	+ 2.68
12.324950	Cape 2	279 7 23.58	46 47 40.41	262	—19.49	+ 5.92
14.318331	Cape 1	280 31 36.40	47 29 6.64	263	—17.17	+ 4.81
14.329283	"	280 32 4.18	47 29 20.97	263	—16.49	+ 3.44
14.522421	Santiago	280 40 14.47	47 33 8.22	267	—20.02	+ 3.53
14.532038	"	280 40 19.50		263	—21.55	
15.295493	Cape 1	281 11 29.82	47 47 55.49	265	—15.34	+ 4.58
15.298458	Cape 2	281 11 39.45	47 48 0.71	263	—12.89	+ 2.60
15.313592	"	281 12 9.87	47 48 14.55	266	—19.11	+ 5.81
15.323250	Cape 1	281 12 35.12	47 48 26.81	265	—17.24	+ 4.42
15.534608	Santiago	281 20 53.70	47 52 21.16	264	—28.95	+ 6.74
16.327195	Cape	281 52 37.43	48 6 48.25	268	—18.66	+ 5.70
16.335931	"		48 6 57.27	268		+ 6.05
17.301841	"	282 30 47.62	—48 23 49.49	269	—15.39	+ 5.35

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Nov. 17.338512	Cape	282 32 11.74		269	—16.47	
17.542119	Santiago	282 39 55.49	—48 27 57.07	269	—24.64	+ 2.37
18.318259	Cape	283 9 44.52	48 40 46.30	271	—18.12	+ 4.96
18.545262	Santiago	283 18 3.81	48 44 21.34	271	—35.40	+ 9.56
19.306291	Cape	283 47 1.08	48 56 29.14	272	—14.78	+ 6.31
19.324727	"	283 47 43.46	48 56 45.77	272	—13.71	+ 6.91
19.342447	"	283 48 22.28		272	—13.98	
19.526554	Santiago	283 55 16.12		272	—12.36	
19.537712	"	283 55 33.25	49 0 6.19	270	—20.14	+ 4.49
20.298313	Cape	284 23 44.33	49 11 39.33	273	—16.03	+ 3.67
20.312796	"	284 24 14.57	49 11 51.54	273	—17.71	+ 4.42
20.327914	"	284 24 50.91		273	—14.69	
20.553316	Santiago	284 32 58.99	49 15 25.53	273	—22.99	+ 4.57
21.310521	Cape	285 0 38.45	49 26 24.96	274	—15.20	+ 5.20
21.325433	"	285 1 9.24	49 26 38.00	274	—16.76	+ 4.96
21.342125	"	285 1 45.37		274	—16.83	
22.337353	"	285 37 22.65	49 40 44.27	275	—21.80	+ 8.54
22.354590	"	285 38 2.02		275	—19.26	
22.543740	Santiago	285 44 41.29	49 43 38.53	277	—23.56	+ 3.35
24.536028	"	286 54 22.66	50 9 38.78	276	—30.78	+ 6.39
26.31198	Cape	287 55 18.16	50 31 18.06	278	—22.57	+ 4.39
26.33066	"	287 55 54.92	50 31 29.58	278	—23.80	+ 6.08
27.30696	"	288 28 48.68	50 42 45.26	279	—23.98	+ 8.32
27.33047	"	288 29 33.10		279	—26.84	
27.54048	Santiago	288 36 27.31	50 45 34.04	282	—34.51	— 1.72
29.31176	Cape	289 35 21.13	51 4 49.73	280	—24.75	+ 6.31
29.33565	"	289 36 5.80		280	—27.23	
30.55715	Santiago	290 16 4.11		281	—25.75	
Dec. 2.31849	Cape	291 13 3.71	51 35 19.27	283	—20.32	+ 5.70
2.34125	"	291 13 45.93		283	—21.91	
2.54202	Santiago	291 20 2.94	51 37 26.70	283	—31.14	+ 7.61
3.31831	Cape	291 44 55.07	51 44 51.74	284	—27.05	+ 5.31
3.32911	"	291 45 16.05	51 44 59.38	284	—24.77	+ 3.18
3.34460	"	291 45 44.62	51 45 8.83	284	—27.72	+ 3.03
3.53935	Santiago	291 51 52.46	51 46 39.08	284	—31.91	+22.20
3.55992	"	291 52 26.16		285	—37.45	
3.56003	"	291 52 31.83	51 47 0.59	286	—31.99	+12.27
4.31389	Cape	292 16 29.01	51 54 3.78	285	—29.96	+ 6.83
4.33605	"	292 17 14.40	51 54 16.34	285	—26.67	+ 6.42
4.54287	Santiago	292 23 40.65	51 56 9.68	286	—32.91	+ 6.07
5.31285	Cape	292 48 5.50	52 3 6.04	287	—24.65	+ 4.85
5.33431	"	292 48 48.04	52 3 15.81	287	—22.62	+ 6.54
6.30326	"	293 19 5.90	52 11 49.21	288	—28.04	+ 3.25
6.30754	"	293 19 16.67	52 11 47.18	289	—25.30	+ 7.50
6.32438	"	293 19 43.38	52 11 58.79	288	—32.07	+ 5.17
6.32800	"	293 19 55.45	52 12 0.96	289	—24.86	+ 4.35
6.34854	"	293 20 33.79		288	—25.11	
6.35494	"	293 20 44.35		289	—26.57	
6.55325	Santiago	293 26 48.31	52 13 50.46	289	—33.43	+11.52
6.57384	"		52 14 7.61	289		+ 5.00
7.55036	"	293 57 46.77	52 22 21.15	290	—39.39	+ 9.40
8.30847	Cape	294 21 25.76	—52 28 45.08	291	—30.71	+ 3.81

Paris M. T. of Observation 1858	Place of Observation	α	δ	Number of Comp. Star	Δα	Δδ
		° ′ ″	° ′ ″		″	″
Dec. 8.33255	Cape	294 22 5.14		291	—36.04	
8.55917	Santiago	294 28 59.09	—52 30 48.53	292	—42.62	+ 4.01
9.55143	"	294 59 40.93	52 38 54.12	292	—36.59	+ 0.57
10.31458	Cape	295 23 9.19	52 44 56.24	294	—34.61	+ 1.95
10.32712	"	295 23 34.77	52 45 2.94	294	—32.09	+ 1.17
10.34029	"	295 23 59.59	52 45 8.07	294	—31.50	+ 2.26
10.35130	"	295 24 16.74		294	—34.60	
10.52539	Santiago	295 30 41.38	52 46 48.94	293	—39.31	+ 6.28
11.31853	Cape	295 54 1.61	52 52 45.94	294	—25.15	+ 1.23
11.34878	"	295 54 53.01		294	—29.17	
11.55938	Santiago	296 1 4.80	52 54 45.46	300	—42.99	— 7.29
12.30595	Cape	296 24 3.24	53 0 11.88	296	—28.91	+ 6.88
12.31831	"	296 24 26.16	53 0 20.06	296	—28.55	+ 4.30
13.55581	Santiago	297 2 23.83	53 9 38.42	295	—22.23	+ 3.52
14.31744	Cape	297 24 59.08	53 15 8.23	298	—30.73	+ 3.29
14.32767	"	297 25 13.25	53 15 11.01	297	—35.11	+ 4.83
14.35668	"	297 26 9.74		297	—31.17	
14.36330	"	297 26 22.17		298	—30.75	
15.55895	Santiago	298 2 7.26	53 24 2.99	297	—48.06	+ 2.54
16.56203	"	298 32 19.10	53 31 4.23	299	—44.47	+ 4.32
19.32976	Cape	299 55 11.85	53 49 59.06	301	—37.07	+ 1.23
19.34409	"	299 55 35.13		301	—39.42	
20.55650	Santiago	300 31 24.36	53 58 6.62	302	—55.79	+ 0.19
21.31330	Cape	300 54 16.50	54 3 0.84	302	—32.96	+ 1.97
21.32952	"	300 54 45.80	54 3 7.25	302	—32.56	+ 1.87
21.56149	Santiago	301 1 16.78	54 4 41.22	302	—54.82	— 1.99
22.31768	Cape	301 24 3.54	54 9 30.07	303	—34.06	+ 1.13
22.33961	"	301 24 44.02	54 9 38.56	303	—32.59	+ 0.95
22.56760	Santiago	301 31 19.79	54 11 13.49	303	—40.57	— 6.57
23.32459	Cape	301 53 58.69	54 15 53.71	304	—28.92	+ 1.73
23.34512	"	301 54 35.10	54 16 2.22	304	—28.19	+ 1.02
24.32131	"	302 23 22.88	54 22 6.89	305	—34.35	+ 4.53
24.34167	"	302 24 1.64	54 22 13.51	305	—31.72	+ 5.55
27.32038	"	303 51 51.23	54 40 36.95	308	—40.07	+ 2.01
27.34108	"	303 52 25.23	54 40 44.18	308	—42.71	+ 2.31
27.57012	Santiago	303 59 16.34	54 42 20.15	306–7	—36.92	—10.40
28.32310	Cape	304 21 29.39	54 46 41.24	309	—36.00	+ 1.08
28.34646	"	304 22 11.90		309	—34.79	
28.57124	Santiago	304 28 48.20	54 48 9.68	309	—36.09	+ 2.09
29.31586	Cape	304 50 41.74	54 52 35.74	311	—39.23	+ 3.40
29.33171	"	304 51 16.86		311	—32.12	
29.55979	Santiago	304 57 49.17	54 54 22.27	312–3	—43.08	—15.87
30.33436	Cape	305 20 43.20	54 58 39.86	312	—38.14	+ 2.62
30.34829	"	305 21 3.13	54 58 45.87	312	—42.86	+ 1.57
30.35422	"		54 58 47.25	312		+ 2.29
30.55880	Santiago	305 26 54.52	55 0 5.65	312	—63.50	— 3.44
30.57658	"		54 59 57.69	312		+10.83
31.32622	Cape	305 49 58.84	55 4 35.18	313	—37.07	— 0.88
31.34421	"	305 50 28.26	55 4 39.50	313	—37.67	+ 0.80
31.56071	Santiago	305 56 41.16	—55 5 49.08	312–10	—47.31	+ 7.64

Paris M. T. of Observation 1859	Place of Observation	α	δ	Number of Comp. Star	Δ α	Δ δ
		° ′ ″	° ′ ″		″	″
Jan. 1.33308	Cape	306 19 32.34	—55 10 24.57	314	—40.72	+ 4.03
1.34183	"		55 10 32.06	314		— 0.38
2.55729	Santiago	306 55 32.95	55 17 58.57	315	—42.76	—21.28
3.56307	"	307 25 5.29	55 23 12.00	316	—47.20	+15.74
3.58347	"		55 23 47.11	316		—12.34
4.32549	Cape	307 47 42.29	55 27 51.49	317	—36.56	+ 1.00
4.33941	"	307 48 0.56	55 27 58.46	317	—43.42	— 1.14
5.32749	"	308 17 8.43	55 33 41.49	318	—41.41	— 2.06
5.33642	"		55 33 43.59	318		— 1.07
5.56881	Santiago	308 24 10.86	55 35 19.42	317	—45.28	—16.58
5.59200	"		55 34 58.82	317		+12.04
6.32873	Cape	308 46 36.88	55 39 23.62	319	—42.15	+ 1.59
6.34175	"	308 47 3.12	55 39 29.44	319	—38.92	+ 0.26
7.56549	Santiago	309 23 0.93	55 46 34.02	320	—44.16	— 2.62
8.56182	"	309 52 19.96	55 52 38.89	321	—46.78	—24.67
10.56311	"	310 51 23.65	56 4 8.39	322	—43.70	—26.07
11.32404	Cape	311 13 47.59	56 8 3.68	323	—46.77	+ 0.31
11.33871	"	311 14 17.25	56 8 7.51	323	—43.11	+ 1.53
11.34569	"		56 8 10.78	323		+ 0.66
11.56671	Santiago	311 20 47.05	56 9 46.64	322–4	—56.99	—18.84
12.33199	Cape		56 13 49.44	325		+ 1.38
13.33141	"	312 13 4.69	56 19 35.81	326	—45.95	— 0.68
13.34791	"	312 13 44.50	56 19 42.70	326	—35.49	— 1.68
20.31958	"	315 40 16.81	57 0 9.91	328	—38.18	— 3.34
20.56512	Santiago	315 47 24.00		327	—49.91	
21.31852	Cape	316 9 55.20	57 6 1.66	329	—43.44	— 2.39
21.32957	"	316 10 12.30	57 6 6.83	329	—46.10	— 3.65
21.33482	"		57 6 8.27	329		— 3.23
21.57642	Santiago	316 17 45.04	57 8 5.68	327	—34.39	—35.18
22.57076	"	316 47 11.78	57 13 8.83	330	—46.21	+15.02
24.32387	Cape	317 39 31.81	57 23 56.51	332	—47.94	— 5.21
24.33599	"	317 39 58.63	57 24 0.42	332	—42.92	— 4.76
24.35405	"	317 40 24.57	57 24 13.69	332	—49.38	— 8.39
24.56212	Santiago	317 46 42.79	57 25 41.62	332	—44.61	—24.56
24.58272	"		57 25 23.07	334		+ 1.41
25.33812	Cape	318 9 58.12	57 30 2.55	333	—43.18	— 5.29
25.57425	Santiago	318 16 45.58		334	—60.23	
26.57324	"	318 46 52.91		334	—50.63	
28.32678	Cape	319 39 44.87	57 48 16.15	335	—41.70	— 6.20
28.34785	"	319 40 22.93	57 48 22.77	335	—40.63	— 5.03
28.57594	Santiago	319 47 6.75		334	—50.06	
29.32330	Cape	320 9 42.05	57 54 24.91	337	—46.43	— 5.37
29.33166	"	320 10 0.52	57 54 26.33	337	—43.07	— 3.68
29.55787	Santiago	320 16 30.68	57 55 15.35	338	—62.44	+31.59
29.57943	"		57 56 15.51	336		—20.53
30.32274	Cape	320 39 50.59	58 0 37.29	339	—48.44	— 4.22
30.33277	"	320 40 10.98	58 0 43.76	339	—46.19	— 6.93
31.33586	"	321 10 33.11	58 7 1.22	341	—44.81	— 6.40
31.57392	Santiago	321 17 53.76	58 8 42.23	343–0	—36.78	—17.24
Feb. 1.57647	"	321 48 9.28		345	—45.36	
1.57647	"	321 48 16.62		343	—38.02	
2.32352	Cape	322 10 51.71	—58 19 42.48	342	—44.52	— 9.14

Paris M. T. of Observation 1859	Place of Observation	α	δ	Number of Comp. Star	$\Delta\alpha$	$\Delta\delta$
		° ′ ″	° ′ ″		″	″
Feb. 2.33020	Cape	322 11 5.56	—58 19 46.24	342	—42.84	—10.37
2.58642	Santiago	322 18 45.70	58 21 28.47	345–8	—50.20	—13.82
2.58642	"	322 18 48.15	58 21 37.72	344	—47.75	—23.07
3.31838	Cape	322 41 10.40	58 26 5.90	344	—42.29	— 7.89
3.32531	"	322 41 20.64	58 26 8.90	344	—44.73	— 8.19
3.58027	Santiago	322 48 55.33	58 27 50.85	346–7	—21.99	—10.99
4.31867	Cape	323 11 35.16	58 32 36.46	346	—47.76	— 8.12
4.32665	"	323 11 48.53	58 32 39.42	346	—49.00	— 7.96
4.56046	Santiago	323 18 56.08		346	—49.82	
5.32129	Cape	323 42 16.59	58 39 12.94	349	—44.69	— 9.73
5.33415	"	323 42 44.72	58 39 20.67	349	—40.20	—12.37
5.55918	Santiago	323 49 20.06	58 41 2.20	350–45	—57.99	—24.75
7.56469	"	324 50 58.45		351	—50.91	
8.32536	Cape	325 14 25.25	58 59 18.46	351	—48.58	— 9.06
21.32089	"	332 1 46.90	60 33 49.94	352	—47.86	— 9.99
21.32628	"	332 1 58.20	60 33 54.71	352	—46.92	—12.23
21.54674	Santiago	332 8 57.59	60 35 35.05	356–3	—50.79	— 9.39
22.32145	Cape	332 33 54.08	60 41 44.39	354	—44.30	—14.10
22.54558	Santiago	332 40 56.92	60 43 7.19	356–5	—53.32	— 9.13
23.54858	"	333 13 31.86		353	—35.53	
24.54393	"	333 45 32.03	60 59 13.05	353	—44.85	+ 0.85
25.31888	Cape	334 10 43.88	61 5 46.15	357	—40.53	—15.09
25.32777	"		61 5 50.04	357		—14.63
26.30820	"	334 42 49.77	61 13 52.40	358	—45.94	—15.06
26.31610	"	334 43 6.12	61 13 56.77	358	—45.04	—15.52
27.31914	"	335 15 56.44	61 22 17.03	359	—40.83	—17.15
28.32881	"	335 48 59.67	61 30 48.03	360	—45.06	—20.45
Mar. 1.30403	"	336 21 4.81	61 39 5.74	361	—47.70	—22.22
1.31416	"	336 21 25.32	61 39 5.96	361	—47.26	—17.26
1.53531	Santiago	336 28 34.91		360	—54.24	
2.30038	Cape	336 54 6.52	61 47 32.30	362	—44.02	—16.36
2.30915	"	336 54 26.91	61 47 35.01	362	—41.08	—14.53
4.31626	"	338 1 22.54	—62 5 13.02	363	—37.19	—22.37

In the next place we proceed to the computation of the perturbations produced by the five large planets, from Venus to Saturn inclusive. The perturbations by Mercury were neglected, as, from the rapid motion of this planet, the intervals of time in the computation of the disturbing forces would require much reduction, with consequent increase of labor, while a rough estimate of the change produced in the comet's geocentric place showed it could not at any time much exceed $0''.1$. To render the integration possible, it was necessary to adopt different intervals of time in the calculation of the disturbing force in different parts of the orbit; the near approach of the comet to Venus, in October, required them to be made as short as one day. The unit of time for the forces given below is however uniformly the same, being ten days. The unit of length is a unit in the seventh decimal place. The forces and perturbations belong to the usual

system of rectangular equatorial co-ordinates; and the constants in the integration have been so taken, that the perturbations are the deviations of the comet from its osculating orbit of October 2.

Washington Mean Noon 1858		X	Y	Z	δx	δy	δz
May	30	+ 2.282	— 4.081	— 2.414	+112.76	+ 59.52	—136.88
June	9	2.812	3.449	2.334	85.88	63.08	115.11
	19	3.199	2.776	2.256	62.38	62.78	95.60
	29	3.376	2.054	2.163	42.59	59.27	78.27
July	9	3.320	1.282	2.013	26.64	53.25	63.03
	19	3.099	— 0.474	1.813	14.44	45.47	49.72
	29	2.756	+ 0.333	1.613	5.76	36.72	38.15
Aug.	8	2.286	1.077	1.434	+ 0.23	27.74	28.12
	18	1.697	1.712	1.295	— 2.58	19.26	18.99
	28	0.998	2.180	1.210	3.23	11.88	11.91
Sept.	2	0.604	2.346	1.192	2.94	8.74	8.91
	7	+ 0.195	2.420	1.197	2.38	6.04	6.29
	12	— 0.200	2.432	1.235	1.69	3.82	4.11
	17	0.564	2.340	1.326	0.99	2.11	2.38
	22	0.831	2.148	1.534	0.43	0.91	1.11
	27	0.898	1.914	1.929	0.10	0.22	0.30
Oct.	2	0.635	1.698	2.668	0.00	0.00	0.00
	7	— 0.062	1.486	4.166	0.06	0.21	0.38
	8	+ 0.123	1.346	4.829	0.08	0.30	0.56
	9	0.322	1.149	5.643	0.10	0.40	0.79
	10	0.621	0.883	6.837	0.12	0.51	1.08
	11	1.163	0.561	8.706	0.14	0.63	1.43
	12	2.122	0.264	11.645	0.14	0.76	1.88
	13	3.894	0.286	16.416	0.12	0.89	2.43
	14	7.326	1.587	24.376	— 0.07	1.02	3.16
	15	14.196	7.291	37.424	+ 0.06	1.17	4.12
	16	26.874	26.084	54.274	0.34	1.40	5.47
	17	39.597	65.057	54.093	0.88	1.91	7.33
	18	31.127	83.863	—16.538	1.80	3.06	9.70
	19	12.813	60.792	+10.765	3.04	5.00	12.23
	20	+ 2.934	36.632	15.493	4.41	7.54	14.67
	21	— 1.001	22.390	13.417	5.82	10.45	16.95
	22	2.572	14.485	10.690	7.23	13.58	19.08
	27	3.790	2.506	3.916	14.00	30.47	28.37
Nov.	1	3.829	+ 0.055	2.246	20.22	47.65	36.45
	6	3.789	— 1.173	1.496	25.87	64.25	43.91
	16	3.605	2.718	0.744	35.42	94.40	58.03
	26	3.216	3.816	0.312	42.52	119.44	72.07
Dec.	6	2.684	4.661	+ 0.029	47.26	138.61	86.62
	16	2.065	5.287	— 0.158	49.93	151.42	102.02
	26	1.410	5.714	0.268	51.01	157.58	118.44
1859							
Jan.	5	0.766	5.957	0.309	51.07	156.92	135.99
	15	— 0.174	6.037	0.286	50.67	149.42	154.69
	25	+ 0.332	5.983	0.209	50.34	135.18	174.48
Feb.	4	0.723	5.831	— 0.092	50.56	114.40	195.30
	14	0.985	5.616	+ 0.053	51.67	87.37	217.00
	24	1.115	5.377	0.205	53.90	54.41	239.42
Mar.	6	+ 1.117	— 5.149	+ 0.353	57.35	+ 15.84	262.41
	16				+ 61.99	— 28.03	—285.79

In forming the normals, the following system of weights was used; the weight being given, not to each observation as published by the observer, but to the result of all the observing in a single night with one comparison star, or with all the stars when they were compared with a single observation of the comet:

The Weight 4 to	The Weight 3 to	The Weight 2 to	The Weight 1 to
Ann Arbor,	Berlin,	Cambridge, Eng.,	Altona,
Bonn,	Cambridge, U. S.,	Christiania,	Armagh,
Cape 1,	Geneva,	Durham,	Batavia,
Greenwich,	Königsberg,	Santiago, Filar Microm.,	Breslau,
Kremsmünster,	Paris,	Vienna,	Copenhagen,
Liverpool,	Göttingen.	Leyden,	Florence,
Pulkova, Mer .Obs.		Pulkova, Ring Microm.,	Markree,
		Cape 2.	Padua,
			Washington,
			Santiago, Ring Microm.

An examination of the Santiago Ring Micrometer Observations shows that when the comet was observed in the northern half of the ring the resulting place is too far to the north, and when in the southern half, too far to the south; which is to be explained by a personal equation in estimating the time of ingress and egress of the comet. I have endeavored to eliminate this source of error by applying a constant correction to the declinations obtained from the northern half of the ring, and the same with a contrary sign to those obtained from the southern half. A comparison of the observations gives $\mp 14''.85$ for this correction. To the right ascensions it appears necessary to add the quantity $+ 2''.35 \sec. \delta$: this was obtained by a comparison with the Cape observations. The normals for convenience are reduced to the nearest Washington Mean Noon, equivalent to $0^{d}.220526$ Paris Mean Time.

	App. α	App. δ	Cor. to Comp. Ephem. $\Delta\alpha$	$\Delta\delta$	Normal formed from Observations between
	° ′ ″	° ′ ″	″	″	
1858. June 14	141 24 27.15	+25 4 45.26	− 2.98	− 5.78	June 7 — June 19
July 13	144 32 38.66	27 47 54.78	− 2.23	+ 0.62	June 28 — July 31
Aug. 11	151 16 58.36	30 57 14.37	− 4.89	+ 5.48	Aug. 4 — Aug. 16
Aug. 23	155 31 25.80	32 43 18.71	− 6.25	+ 8.59	Aug. 17 — Aug. 28
Sept. 5	162 9 37.57	34 58 27.81	− 8.42	+12.96	Aug. 30 — Sept. 11
Sept. 17	172 46 57.63	36 27 32.59	−12.67	+14.44	Sept. 12 — Sept. 22
Sept. 28	192 7 59.94	32 26 23.74	−12.43	+14.21	Sept. 23 — Oct. 3
Oct. 8	221 13 0.02	+10 44 14.24	− 5.19	+ 8.99	Oct. 4 — Oct. 14
Oct. 19	250 27 5.84	−23 43 25.28	+ 0.41	+ 0.50	Oct. 15 — Oct. 25
Nov. 1	269 34 10.88	41 1 15.00	− 7.15	+ 3.30	Oct. 26 — Nov. 7
Nov. 16	281 48 26.58	48 4 54.63	−16.49	+ 4.70	Nov. 9 — Nov. 22
Dec. 1	290 37 35.66	51 24 31.43	−25.62	+ 5.38	Nov. 24 — Dec. 6
Dec. 16	298 22 14.09	53 28 43.21	−34.40	+ 2.16	Dec. 8 — Dec. 24
1859. Jan. 3	307 15 6.95	55 21 28.01	−40.33	+ 0.54	Dec. 27 — Jan. 13
Jan. 30	320 36 49.56	58 0 1.20	−44.14	− 6.47	Jan. 20 — Feb. 8
Feb. 26	334 40 0.58	−61 13 10.20	−43.65	−16.18	Feb. 21 — Mar. 4

The following remarks must be made with regard to the composition of these normals:

June 14. The right ascension is the mean of four Berlin observations; the rest are so discordant that no confidence can be placed in them.

July 13. This normal is formed from the Berlin, Cambridge, and Ann Arbor observations, the others being rejected. The Washington observations, although more concordant at this time than they are generally, yet differ from the observations which should be considered the best, and on trial it has proved impossible to satisfy them along with the other normals.

Oct. 19. The right ascension of this normal has proved most refractory; when formed from all the material, it could not possibly be represented within $2''.5$, and much experimenting showed that a curve drawn through the adjacent normals would leave this one distant from it by about that quantity. This difference seeming altogether too large to be admitted in a normal having so much weight, some means must be adopted for ameliorating it.

As a more careful scrutiny of the observations showed that those made with small telescopes, especially those made at the Cape with the small instrument, had produced this deviation, I reluctantly set them aside; and the right ascension given above is the result of the Berlin, Bonn, Cambridge, U. S., and Ann Arbor observations.

By subtracting the reductions given below, we obtain the co-ordinates of the comet referred to the mean equinox and equator of 1858.0, and freed from perturbations.

		Aberration.		Reduction from 1858.0.		Perturbations.							
1858		$\Delta\alpha$	$\Delta\delta$	$\Delta\alpha$	$\Delta\delta$	$\Delta\alpha$	$\Delta\delta$	α 1858.0			δ 1858.0		
		$''$	$''$	$''$	$''$	$''$	$''$	°	′	″	°	′	″
June	14	− 2.20	− 5.26	+31.32	− 3.43	−0.86	−0.71	141	23	58.89	+25	4	54.66
July	13	8.58	4.72	37.80	5.93	0.51	0.47	144	32	9.95	27	48	5.90
Aug.	11	13.36	5.78	43.55	9.07	0.26	0.28	151	16	28.43	30	57	29.50
Aug.	23	15.79	6.25	45.52	10.68	0.16	0.19	155	30	56.23	32	43	35.83
Sept.	5	20.37	5.51	47.12	12.86	0.10	0.12	162	9	10.92	34	58	46.30
Sept.	17	28.74	− 0.22	47.06	15.65	0.04	0.05	172	46	39.35	36	27	48.51
Sept.	28	38.30	+15.97	43.23	18.90	0.00	0.00	192	7	55.01	32	26	26.67
Oct.	8	35.41	38.05	40.05	19.27	−0.01	0.02	221	12	55.39	+10	43	55.48
Oct.	19	27.95	30.56	48.53	14.35	+0.04	0.42	250	26	45.22	−23	43	41.07
Nov.	1	22.57	15.90	60.30	8.58	0.52	1.17	269	33	32.63	41	1	21.15
Nov.	16	20.43	9.26	69.54	− 3.84	1.11	1.41	281	47	36.36	48	4	58.64
Dec.	1	21.23	6.56	76.45	+ 0.11	1.53	1.43	290	36	38.91	51	24	36.67
Dec.	16	23.27	5.43	82.19	3.83	1.79	1.38	298	21	13.38	53	28	51.09
1859													
Jan.	3	26.29	5.16	88.00	8.24	1.92	1.23	307	14	3.32	55	21	40.18
Jan.	30	30.51	6.31	94.21	14.70	1.72	0.94	320	35	44.14	58	0	21.27
Feb.	26	−35.09	+ 8.86	+96.56	+20.85	+0.90	−0.61	334	38	58.21	−61	13	39.30

In forming equations of condition from these normals, it will be advantageous to use residuals from elements nearer the truth than those of Searle. The following elements, computed from three provisional normals, embracing the whole period of the comet's apparition, will serve this purpose.

$$T = 1858, \text{ Sept. } 29.971007, \text{ Paris Mean Time.}$$

$$\left.\begin{aligned} \omega &= 129^\circ\ 6'\ 39''.40, \\ \Omega &= 165\ 19\ 10.67, \\ i &= 116\ 58\ 10.87, \end{aligned}\right\} \text{Mean Equinox and Ecliptic, 1858.0.}$$

$$\begin{aligned} \varphi &= 85\ \ 2\ 43.72, \\ \log q &= 9.7622760, \\ \log a &= 2.18982, \\ P &= 1926^{y}.3. \end{aligned}$$

The places of the Sun used will be taken from Hansen and Olufsen's *Tables du Soleil*, substituting, however, the Pulkova constants of nutation and aberration. A comparison of the Greenwich observations of the Sun, for 1858–59, shows a pretty good representation of observation by these tables; and the small differences that remain may be much modified by the introduction of corrections peculiar to the observer and the instrument. Knowing the difficulty that attends the consideration of this matter, I do not propose to inquire further into it.

The following are the equations of condition that result from the above normals. The logarithms of the coefficients are given instead of the coefficients themselves, and the variations of the elements are supposed to be expressed in seconds of arc, $0^{d}.0001$ in δT being equivalent to $1''$, and 0.00001 in $\delta \log q$ and δe; the right hand members are $\Delta\alpha \cos \delta$ and $\Delta\delta$.

Equations from the Right Ascensions.

$\delta \log q$	δe	δT	$\delta \omega$	δi	$\delta \Omega$		Weight.
—9.8662	+9.3875	+8.7810	—9.2176	—9.2190	+9.8566	= +1.17	0.12
—9.8551	+9.0785	+8.7925	—9.2008	—9.3646	+9.7651	= +0.92	0.36
—9.8000	+8.5855	+8.7743	—9.1247	—9.5208	+9.6436	= +0.18	0.62
—9.7273	+8.2393	+8.7146	—9.0325	—9.5931	+9.5496	= +0.75	0.70
—9.4914	—7.3061	+8.3722	—8.7217	—9.6767	+9.3190	= +1.63	1.73
+9.1499	—8.3465	—8.8318	+8.7980	—9.7498	—8.6312	= +0.55	2.57
+9.9901	—8.1188	—9.4913	+9.4492	—9.7556	—9.6671	= +0.33	2.09
+0.3761	+9.1757	—9.7442	+9.5448	—9.4468	—9.8318	= +0.38	2.27
+0.4981	+9.5597	—9.4693	—9.1112	+7.5105	—8.7024	= +2.07	1.19
+0.4498	+9.6413	—8.7919	—9.6233	—8.8691	+9.5399	= +1.58	0.79
+0.3823	+9.6892	+7.8946	—9.6984	—9.2986	+9.6547	= +2.71	0.72
+0.3199	+9.7175	+8.3883	—9.5057	—9.4785	+9.6702	= +2.99	0.46
+0.2584	+9.7263	+8.4389	—9.6900	—9.5929	+9.6637	= +2.24	0.62
+0.1787	+9.7084	+8.4115	—9.6505	—9.6946	+9.6418	= +2.43	0.60
+0.0237	+9.6084	+8.3032	—9.5379	—9.8103	+9.5822	= +2.44	0.67
+9.7381	+9.2739	+8.1126	—9.2732	—9.8974	+9.4698	= +0.10	0.41

Equations from the Declinations.

$\delta \log q$	δe	δT	$\delta \omega$	δi	$\delta \Omega$		Weight.
+0.4299	+9.8023	—8.9052	+9.9005	—8.6591	—9.2630	= +4".59	0.29
+0.4008	+9.6194	—8.9529	+9.8126	—8.7826	—9.2068	= +3.61	0.47
+0.4014	+9.3636	—9.0475	+9.7291	—8.8411	—9.2140	= +1.61	0.60
+0.4094	+9.1945	—9.0967	+9.6871	—8.8276	—9.2258	= +1.72	0.70
+0.4223	+8.9104	—9.1314	+9.6160	—8.7090	—9.2238	= +2.81	1.71
+0.4444	+8.5223	—8.9832	+9.4363	+7.0656	—9.0928	= +2.21	2.57
+0.5294	+7.9961	+9.2695	—9.1089	+9.0218	+9.0810	= +1.00	2.04
+0.6902	—9.1747	+9.9580	—9.9980	+8.5811	+9.7179	= +0.25	2.24
+0.6299	—9.4551	+9.8831	—9.9774	+8.1283	—8.3321	= —1.12	1.15
+0.4712	—9.1778	+9.5374	—9.7775	+9.3709	—9.3505	= —0.40	0.79
+0.3824	—8.4943	+9.2865	—9.7031	+9.5341	—9.1944	= +0.26	0.72
+0.3437	+8.7707	+9.1361	—9.6986	+9.5768	—8.8537	= +1.90	0.44
+0.3297	+9.1485	+9.0385	—9.7200	+9.5823	+7.6943	= +0.53	0.62
+0.3304	+9.3778	+8.9605	—9.7595	+9.5616	+8.9410	= +2.06	0.60
+0.3495	+9.6005	+8.8879	—9.8280	+9.4755	+9.3028	= +1.97	0.60
+0.3754	+9.7620	+8.8403	—9.8951	+9.2523	+9.4895	= +1.68	0.39

The operations were carried through with logarithms of five decimal places; the want of breadth in the page has compelled the omission of the last figure in the above coefficients. The resulting normal equations are—

$\delta \log q$	δe	δT	$\delta \omega$	δi	$\delta \Omega$		
+211.720	+6.2418	+9.9751	—16.7517	—0.8780	+1.5523	—81".633	=0
+ 6.2418	+1.8262	—0.9109	— 0.0865	—0.6452	+0.4804	— 9.5672	=0
+ 9.9751	—0.9109	+3.8401	— 4.1916	+1.0919	+2.3800	+ 2.7880	=0
— 16.7517	—0.0865	—4.1916	+ 7.2621	—0.8845	—3.3057	— 0.5563	=0
— 0.8780	—0.6452	+1.0919	— 0.8845	+3.5740	+0.0632	+ 5.3321	=0
+ 1.5523	+0.4804	+2.3800	— 3.3057	+0.0632	+3.6073	— 2.6113	=0

The solution of these gives—

$\delta \log q = +0''.44$, $\delta e = +2''.99$, $\delta T = -0''.36$, $\delta\omega = +1''.81$, $\delta i = -0''.32$, $\delta\Omega = +2''.04$,

and the sum of the squares of the residuals is reduced from 87.378 to 13.547, making the probable error of a normal of the weight unity, $\pm 0''.487$. Adopting this value, the elements with their probable errors are (which elements it will be remembered are the osculating of Oct. 2):

$T =$ 1858, Sept. 29.970971 $\pm$ 0^d.0000860 Paris Mean Time.

$\omega =$	129° 6′ 41″.21	$\pm$ 0″.348	Mean Equinox and Ecliptic 1858.0
$\Omega =$	165 19 12.71	$\pm$ 0.611	Mean Equinox and Ecliptic 1858.0
$i =$	116 58 10.55	$\pm$ 0.290	Mean Equinox and Ecliptic 1858.0
$\varphi =$	85 3 55.22	$\pm$ 19.10	
$\log q =$	9.7622804	$\pm$ 0.000000616	
$\log a =$	2.19331		
$P =$	1949.7 years.	$\pm$ 6^y.25	

The normals are represented by these elements with the following residuals (Obs. — Cal.):

	$\Delta\alpha\cos\delta$	$\Delta\delta$		$\Delta\alpha\cos\delta$	$\Delta\delta$
	$''$	$''$		$''$	$''$
June 14	−0.43	+0.39	Oct. 19	−0.17	−0.11
July 13	−0.07	+0.35	Nov. 1	−0.96	+0.49
Aug. 11	−0.40	−0.89	Nov. 16	+0.11	+0.70
Aug. 23	+0.30	−0.49	Dec. 1	+0.39	+1.97
Sept. 5	+1.23	+0.93	Dec. 16	−0.32	+0.27
Sept. 17	+0.32	+0.60	Jan. 3	+0.00	+1.42
Sept. 28	+0.08	−0.44	Jan. 30	+0.41	+0.73
Oct. 8	−0.66	−0.39	Feb. 26	−1.21	−0.22

These residuals, although they appear quite small, do not indicate a completely satisfactory solution. For the probable error derived from them is much larger than that obtained from the consideration of the observations themselves. The latter quantity is $\pm 0''.27$, while the former, as stated above, is $\pm 0''.487$. The principal cause of this difference is doubtless to be sought in the small systematic errors of the observations which arise from the idiosyncrasy of the observer in selecting the proper point to be observed, influenced perhaps, to some degree by the size of the instrument he used. In Vol. III., p. 329, of the Annals of Harvard College Observatory, the opinion is expressed that the observations have a tendency to place the comet too near the Sun, and the smaller the telescope the nearer the Sun. Let us see whether the observations confirm this supposition. Taking the comparisons in declination of the best observations which go to form our normal of Sept. 17, when the effect of such a tendency lies almost wholly in declination, and arranging them under the heads of the different observatories and in the order of the size of the telescopes, we have the following table. The numbers beneath the names of the observatories denote the aperture of the telescope in inches.

	Ann Arbor. 12.5	Berlin. 9.6	Liverpool. 8.5	Königsberg. 6.25	Bonn. 6.0	Kremsmünster. 5.9	Pulkova. 5.8	Paris. 4.8	Geneva. 4.25
	$''$	$''$	$''$	$''$	$''$	$''$	$''$	$''$	$''$
Sept. 12	+14.33	......	+15.63	+22.06	+13.93	+14.42	+13.41	+17.28	+10.28
13	+14.72	+13.93	......	+12.59	......	+15.48	......	+19.45	+15.92
14	......	......	......	+13.57	......	+13.63	......	+17.69	+12.28
15	......	+16.04	+16.06	+21.30	......	......	......	......	+ 7.63
16	+14.47	......	......	+10.53	+11.66	+15.72	+11.15	......	......
17	+16.65	......	......	+18.84	......	+16.64	+14.35	......	......
18	+17.32	......	+12.06	+10.13	......	......	+ 9.83	+25.92	......
19	+12.55	......	......	......	......	+17.81	......	+ 7.51	+13.14
20	+14.10	+13.48	......	......	......	+13.83	......	+13.09	......
21	......	+15.17	+18.13	+11.08	+14.28	......	......	......	+18.35
22	......	......	......	......	+12.90	+15.55	......	......	......
Mean,	+14.88	+14.66	+15.47	+15.01	+13.19	+15.39	+12.19	+16.82	+12.93

The existence of systematic error seems pretty well made out between the different observatories; and the Bonn, Pulkova, and Geneva observations made with small telescopes, do certainly place the comet nearer the Sun than the others. But the observatory which places the comet farthest to the north is Paris, with a very small telescope. Also Kremsmünster and Königsberg, with much smaller telescopes, put the comet farther from the Sun than Ann Arbor and Berlin. These facts militate strongly against this supposition. The quantity used in forming the normal was $+14''.44$, and the preceding elements give $+13''.84$ for the same quantity, from which it may be judged how well each of the above observations is satisfied.

Again, if this hypothesis were sufficient to account for the systematic errors, we should have almost perfect agreement in the right ascensions. Let us see whether this is the case.

	Ann Arbor.	Berlin.	Liverpool.	Königsberg.	Bonn.	Kremsmünster.	Pulkova.	Paris.	Geneva.
	″	″	″	″	″	″	″	″	″
Sept. 12	−7.62	......	−11.69	−7.21	−13.13	−10.38	−18.72	−4.79	−9.74
13	−11.36	−18.00	......	−9.26	......	−12.00	......	−9.48	−14.06
14	......	......	......	−7.87	......	−6.96	......	−9.41	−9.83
15	......	−17.88	−8.83	−11.24	......	......	......	......	−16.48
16	−14.63	......	......	−12.52	−7.51	−10.77	−10.14	......	......
17	−13.47	......	......	−13.64	......	−15.02	−13.25	......	......
18	−13.13	......	−11.09	−5.55	......	......	−10.07	−15.38	......
19	−17.26	......	......	......	......	−13.13	......	−18.14	−15.42
20	−16.22	−9.98	......	......	......	−10.89	......	−20.02	......
21	......	−11.95	−9.02	−9.39	−14.82	......	......	......	−8.18
22	......	......	......	......	−19.26	−12.99	......	......	......
Mean,	−13.38	−14.45	−10.16	−9.59	−13.68	−11.52	−13.05	−12.87	−12.29

Systematic error is not quite so manifest here as in the declinations, the observations not agreeing so well among themselves, but it undoubtedly exists in considerable quantity. The quantity used for the normal of Sept. 17 was $-12''.67$, and the orbit found gives $-13''.07$.

We will make one more trial; about Oct. 8, the effect, according to the hypothesis, took place wholly in the direction of right ascension. The scheme of observations stands thus:

	Ann Arbor.	Paris. 12.6	Berlin.	Liverpool.	Königsberg.	Bonn.	Kremsmünster.	Pulkova.	Geneva.	Greenwich. 3.75
	″	″	″	″	″	″	″	″	″	″
Oct. 4	.....	.....	−7.20	−2.82	......	.....	......	......	−16.18	.....
5	−7.94	.....	+1.71	.....	−7.91	−7.65	......	......	−10.96	−9.49
6	.....	.....	−4.30	.....	......	.....	−11.01	−10.98	−11.00	.....
7	.....	.....	−4.97	.....	......	.....	−6.66	......	−8.19	.....
8	.....	−2.80	.....	−8.49	......	−4.85	−2.40	......	......	.....
9	−0.09	−4.29	.....	.....	−12.66	.....	......	......	−6.94	−0.68
10	.....	.....	.....	.....	......	.....	−6.49	......	......	.....
11	.....	.....	.....	.....	......	.....	−4.78	......	......	−2.05
12	.....	.....	.....	.....	......	.....	......	−4.17	......	.....
13	−1.20	.....	.....	+4.11	......	.....	......	......	−5.28	.....
14	+5.45	.....	.....	.....	......	.....	−3.73	......	−4.39	.....

The observations are too scattered to establish anything with certainty, but the systematic errors seem to be larger than before, and, Greenwich excepted, the observations with the small telescopes place the comet farther from the Sun than those with the large telescopes. The same thing is probably true of the observations during the rest of October, but as the northern observations here begin to fail us, we can make no comparison.

It would be very difficult, perhaps impossible, to arrive at a satisfactory explanation of these systematic errors and to assign their numerical values, consequently I shall not undertake any discussion of them. If, however, this hypothesis should be adopted, and a correction varying inversely as the size of the telescope should be applied to the observations, removing the comet from the Sun a space ranging from $1''$ to $3''$, the effect would be to diminish the period of revolution by about 25 or 30 years. With regard to this, the most interesting element of the orbit, we may state with confidence, I think, that it is not less than 1900 years, and cannot exceed 1975 years.

Lastly, we have settled by this discussion, that there is not the slightest indication that any other force than gravity influenced the motion of the center of gravity of the comet. For although, on comparing our final orbit with observations made at a particular observatory, we should observe small but well marked deviations, yet another observatory will be found whose observations, entitled to equal confidence, indicate a deviation at the same time in an opposite direction.

MEMOIR No. 7.

On the Derivation and Reduction of Places of the Fixed Stars.

(Extracted from the Star Tables of the American Ephemeris, 1869.)

The coordinates of the stars are affected by three distinct causes; first, by the motion of the earth's axis and the equinox, which produces precession and nutation; second, by the motion of the star itself and of the solar system in space, the combined effect of which is denoted as proper motion; third, by the motion of light itself, the effect of which is called aberration.

1. Let us first consider the effect of procession alone. If α and δ denote the right ascension and declination of a star at any time, its rectangular coordinates will be, its distance being assumed as unity,

$$\left.\begin{aligned} x &= \cos\delta\cos\alpha\,, \\ y &= \cos\delta\sin\alpha\,, \\ z &= \sin\delta\,. \end{aligned}\right\} \qquad (1)$$

To pass to any new system, we shall have the known equations

$$\left.\begin{aligned} x' &= ax + by + cz\,, \\ y' &= a'x + b'y + c'z\,, \\ z' &= a''x + b''y + c''z\,. \end{aligned}\right\} \qquad (2)$$

But in the case where we wish to obtain the differentials of x, y, z for an infinitesimal time dt, a, b' and c'' are each unity, being the cosines of angles infinitely small; and all the other constants will contain dt as a factor. Hence we may write

$$\left.\begin{aligned} \frac{dx}{dt} &= by + cz\,, \\ \frac{dy}{dt} &= a'x + c'z\,, \\ \frac{dz}{dt} &= a''x + b''y\,. \end{aligned}\right\} \qquad (3)$$

The equation $x^2 + y^2 + z^2 = 1$ gives us $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt} = 0$. Substituting in this the above values of $\dfrac{dx}{dt}$, etc., there result these three equations of condition between the six remaining constants

$$b + a' = 0\,, \quad c + a'' = 0\,, \quad c' + b'' = 0\,. \qquad (4)$$

Hence,

$$\left.\begin{aligned}\frac{dx}{dt}&=by+cz\,,\\ \frac{dy}{dt}&=-bx+c'z\,,\\ \frac{dz}{dt}&=-cx-c'y\,.\end{aligned}\right\}\qquad(5)$$

It belongs to Celestial Mechanics to deduce the values of the three remaining coefficients of these equations. When precession alone is considered, $c'=0$, and $-b$ and $-c$ are the quantities usually denoted by m and n. Thus we have, the unit of t being one year,

$$\left.\begin{aligned}\frac{dx}{dt}&=-my-nz\,,\\ \frac{dy}{dt}&=mx\,,\\ \frac{dz}{dt}&=nx\,.\end{aligned}\right\}\qquad(6)$$

If the values of x, y and z are now substituted in these equations, we find that

$$\left.\begin{aligned}\frac{d\alpha}{dt}&=m+n\sin\alpha\tan\delta\,,\\ \frac{d\delta}{dt}&=n\cos\alpha\,.\end{aligned}\right\}\qquad(7)$$

m and n are functions of t which admit of being expressed by power series.

Differentiating (7) and always eliminating $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$ by means of the primitive equations, we obtain

$$\left.\begin{aligned}\frac{d^2\alpha}{dt^2}&=\frac{dm}{dt}+\frac{n^2}{2}\sin 2\alpha+\left(\frac{dn}{dt}\sin\alpha+mn\cos\alpha\right)\tan\delta+n^2\sin 2\alpha\tan^2\delta\,,\\ \frac{d^2\delta}{dt^2}&=-mn\sin\alpha+\frac{dn}{dt}\cos\alpha-n^2\sin^2\alpha\tan\delta\,,\\ \frac{d^3\alpha}{dt^3}&=\frac{mn^2}{2}+\tfrac{3}{2}mn^2\cos 2\alpha+\tfrac{3}{2}n\frac{dn}{dt}\sin 2\alpha\\ &\quad+\left[(2n^2-m^2+3n^2\cos 2\alpha)n\sin\alpha+\left(2m\frac{dn}{dt}+n\frac{dm}{dt}\right)\cos\alpha\right]\tan\delta\\ &\quad+\left[3mn^2\cos 2\alpha+3n\frac{dn}{dt}\sin 2\alpha\right]\tan^2\delta\\ &\quad+2n^3\sin\alpha\,(1+2\cos 2\alpha)\tan^3\delta\,,\\ \frac{d^3\delta}{dt^3}&=-\left(2m\frac{dn}{dt}+n\frac{dm}{dt}\right)\sin\alpha-(m^2+n^2\sin^2\alpha)n\cos\alpha\\ &\quad-\left[\tfrac{3}{2}mn^2\sin 2\alpha+3n\frac{dn}{dt}\sin^2\alpha\right]\tan\delta\\ &\quad-3n^3\sin^2\alpha\cos\alpha\tan^2\delta\,.\end{aligned}\right\}\qquad(8)$$

In writing these equations, it has been assumed that $\frac{d^3m}{dt^3}$ and $\frac{d^3n}{dt^3}$ vanish.

The right ascension and declination of a star, as far as regards precession, are then found by the formulas

$$\left.\begin{aligned} \alpha &= \alpha_0 + \left(\frac{d\alpha}{dt}\right)_0 t + \tfrac{1}{2}\left(\frac{d^2\alpha}{dt^2}\right)_0 t^2 + \tfrac{1}{6}\left(\frac{d^3\alpha}{dt^3}\right)_0 t^3 + \ldots, \\ \delta &= \delta_0 + \left(\frac{d\delta}{dt}\right)_0 t + \tfrac{1}{2}\left(\frac{d^2\delta}{dt^2}\right)_0 t^2 + \tfrac{1}{6}\left(\frac{d^3\delta}{dt^3}\right)_0 t^3 + \ldots \end{aligned}\right\} \quad (9)$$

2. Let us next consider the effect of proper motion. If the values of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$ for any star are obtained from observation for a certain epoch, we may compute the functions $m + n \sin\alpha \tan\delta$ and $n\cos\alpha$, and subtract them from these quantities, the remainders μ and μ' are the effect of proper motion in right ascension and declination at that epoch. But, to deduce the values of μ and μ' for any time in general, we may adopt the assumption that the proper motion is uniform on the arc of a great circle, and on this supposition derive the rigorous values of the differential coefficients of α and δ with respect to the time.

Considering now the effect of proper motion only, let

ρ denote the velocity of the star's motion on the arc of a great circle,

χ the angle of position of this arc,

α' and δ' the right ascension and declination of the star at the end of the time t.

The consideration of the spherical triangle formed by the pole of the equator and the two positions of the star will give these equations,

$$\left.\begin{aligned} \sin\delta' &= \sin\delta\cos(\rho t) + \cos\delta\sin(\rho t)\cos\chi, \\ \cos\delta'\cos(\alpha' - \alpha) &= \cos\delta\cos(\rho t) - \sin\delta\sin(\rho t)\cos\chi, \\ \cos\delta'\sin(\alpha' - \alpha) &= \sin(\rho t)\sin\chi. \end{aligned}\right\} \quad (10)$$

Eliminating ρ and χ by means of the equations

$$\rho\sin\chi = \mu\cos\delta, \quad \rho\cos\chi = \mu',$$

we derive from the first and third of the preceding equations the following values of α' and δ' in series arranged according to the powers of t:

$$\left.\begin{aligned} \alpha' &= \alpha + \mu t + \mu\mu' \tan\delta \,.\, t^2 - \tfrac{1}{3}[\mu^3 \sin^2\delta - \mu\mu'^2(1 + 3\tan^2\delta)]\, t^3 + \ldots, \\ \delta' &= \delta + \mu' t - \tfrac{1}{4}\mu^2 \sin 2\delta \,.\, t^2 - \tfrac{1}{6}\mu^2\mu'(1 + 2\sin^2\delta)\, t^3 + \ldots \end{aligned}\right\} \quad (11)$$

3. In order to have the combined effect of precession and proper motion, α' and δ' should be substituted for α and δ in the series which give the effect of precession. Hence, we obtain

$$\left.\begin{aligned}\frac{d\alpha}{dt} &= m + n\sin\alpha\tan\delta + \mu\,,\\ \frac{d\delta}{dt} &= n\cos\alpha + \mu'\,;\end{aligned}\right\}\qquad(12)$$

and, μ and μ' being considered as variable quantities,

$$\left.\begin{aligned}\frac{d\mu}{dt} &= n\mu\cos\alpha\tan\delta + n\mu'\sin\alpha\sec^2\delta + 2\mu\mu'\tan\delta\,,\\ \frac{d\mu'}{dt} &= -n\mu\sin\alpha - \tfrac{1}{2}\mu^2\sin 2\delta\,.\end{aligned}\right\}\qquad(13)$$

It may be useful to note the rate of variation of the angle of position χ through the effects of precession and proper motion; it is

$$\frac{d\chi}{dt} = n\sin\alpha\sec\delta + \mu\sin\delta\,.\qquad(14)$$

By differentiating the values of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$, and eliminating $\frac{d\alpha}{dt}$, $\frac{d\delta}{dt}$, $\frac{d\mu}{dt}$ and $\frac{d\mu'}{dt}$ by means of their values just given, we obtain

$$\left.\begin{aligned}
\frac{d^2\alpha}{dt^2} = {} & \frac{dm}{dt} + \frac{n^2}{2}\sin 2\alpha + 2n\mu'\sin\alpha\\
& + \left[\frac{dn}{dt}\sin\alpha + (m+2\mu)\,n\cos\alpha + 2\mu\mu'\right]\tan\delta\\
& + 2n\sin\alpha\,(n\cos\alpha + \mu')\tan^2\delta\,,\\
\frac{d^2\delta}{dt^2} = {} & -mn\sin\alpha + \frac{dn}{dt}\cos\alpha - 2n\mu\sin\alpha - \frac{\mu^2}{2}\sin 2\delta\\
& - n^2\sin^2\alpha\tan\delta\,,\\
\frac{d^3\alpha}{dt^3} = {} & \frac{mn^2}{2} + 2\mu\mu'^2 + 3\frac{dn}{dt}\mu'\sin\alpha + 3n\mu'(m+2\mu)\cos\alpha\\
& + \tfrac{3}{2}(m+2\mu)\,n^2\cos 2\alpha + \tfrac{3}{2}n\frac{dn}{dt}\sin 2\alpha - 2\mu^2\sin^2\delta\\
& + \left[\begin{aligned}&(2n^2 - m^2 - 6\mu^2 - 3m\mu + 3n^2\cos 2\alpha)\,n\sin\alpha\\ &+\left(2m\frac{dn}{dt} + n\frac{dm}{dt} + 3\frac{dn}{dt}\mu\right)\cos\alpha + 6n^2\mu'\sin 2\alpha\\ &\qquad + 6\mu'^2\end{aligned}\right]\tan\delta\\
& + \left[\begin{aligned}&6\mu\mu'^2 + 3\frac{dn}{dt}\mu'\sin\alpha + (3m+12\mu)\,n\mu'\cos\alpha\\ &+ 3n\frac{dn}{dt}\sin 2\alpha + (3m+6\mu)\,n^2\cos 2\alpha\end{aligned}\right]\tan^2\delta\,,\\
& + [(2n^2 + 6\mu'^2)\,n\sin\alpha + 6n^2\mu'\sin 2\alpha + 4n^3\sin\alpha\cos 2\alpha]\tan^3\delta\,,\\
\frac{d^3\delta}{dt^3} = {} & -\mu^2\mu' - (2m+3\mu)\frac{dn}{dt}\sin\alpha - (m^2 + 3m\mu + 3\mu^2)\,n\cos\alpha\\
& - n\frac{dm}{dt}\sin\alpha - n^2\sin^2\alpha\cos\alpha - 3n^2\mu'\sin^2\alpha - 2\mu^2\mu'\sin^2\delta\\
& - \left[6n\mu\mu'\sin\alpha + \tfrac{3}{2}(m+2\mu)n^2\sin 2\alpha + 3n\frac{dn}{dt}\sin^2\alpha\right]\tan\delta\\
& - 3(n\cos\alpha + \mu')\,n^2\sin^2\alpha\tan^2\delta\,.
\end{aligned}\right\}\qquad(15)$$

The values of α and δ, computed by means of Maclaurin's Theorem, using the above values of the differential coefficients, will give the mean place of the star. The last term of $\frac{d^2\alpha}{dt^2}$ and also that of $\frac{d^2\delta}{dt^2}$ are nearly always insensible.

The expressions for $\frac{d^3\alpha}{dt^3}$ and $\frac{d^3\delta}{dt^3}$ above are too complicated for use in computation; hence, if their values are wanted, it will be much easier to compute the values of the second differential coefficients for 50 years before and after the epoch, and divide the differences of those by 100 for the value of the third differential coefficients at the epoch.

4. We have next to consider the effect of nutation. Resuming equations (5), putting for x, y and z their values from (1) in terms of α and δ, and writing $\Delta\alpha$ and $\Delta\delta$ instead of $\frac{d\alpha}{dt}$ and $\frac{d\delta}{dt}$, we obtain.

$$\left.\begin{aligned}\Delta\alpha &= -b - c\sin\alpha\tan\delta + c'\cos\alpha\tan\delta,\\ \Delta\delta &= -c\cos\alpha - c'\sin\alpha.\end{aligned}\right\}\tag{16}$$

Changing the notation so as to correspond with that usually employed in this subject, we make

$$b = -mA' - E,\quad c = -nA',\quad c' = B,\tag{17}$$

where A' is the quantity usually denoted by A with the term τ, the fraction of the year, omitted. Then

$$\left.\begin{aligned}\Delta\alpha &= (m + n\sin\alpha\tan\delta)A' + B\cos\alpha\tan\delta + E,\\ \Delta\delta &= A'n\cos\alpha - B\sin\alpha.\end{aligned}\right\}\tag{18}$$

These formulas give the effect of nutation when terms multiplied by the squares and products of A', B and E are neglected.

The following formulas contain those which involve the squares and products of A' and B, still neglecting the square of E and its products with A' and B as of no moment:

$$\left.\begin{aligned}\Delta^2\alpha &= \tfrac{1}{2}\frac{d^2\Delta\alpha}{dA'^2}A'^2 + \frac{d^2\Delta\alpha}{dA'dB}A'B + \tfrac{1}{2}\frac{d^2\Delta\alpha}{dB^2}B^2,\\ \Delta^2\delta &= \tfrac{1}{2}\frac{d^2\Delta\delta}{dA'^2}A'^2 + \frac{d^2\Delta\delta}{dA'dB}A'B + \tfrac{1}{2}\frac{d^2\Delta\delta}{dB^2}B^2.\end{aligned}\right\}\tag{19}$$

We have from (18)

$$\left.\begin{aligned}\frac{d\,.\,\Delta\alpha}{dA'} &= m + n\sin\alpha\tan\delta, & \frac{d\,.\,\Delta\delta}{dA'} &= n\cos\alpha,\\ \frac{d\,.\,\Delta\alpha}{dB} &= \cos\alpha\tan\delta, & \frac{d\,.\,\Delta\delta}{dB} &= -\sin\alpha.\end{aligned}\right\}\tag{20}$$

Differentiating these again with respect to A' and B, and eliminating $\frac{d\alpha}{dA'}$, $\frac{d\alpha}{dB}$, etc., which are the same as $\frac{d\,.\,\Delta\alpha}{dA'}$, $\frac{d\,.\,\Delta\alpha}{dB}$, etc., we obtain

$$\left.\begin{aligned}
\frac{d^2\Delta\alpha}{dA'^2} &= \frac{n^2}{2}\sin 2\alpha + mn\cos\alpha\tan\delta + n^2\sin 2\alpha\tan^2\delta,\\
\frac{d^2\Delta\alpha}{dA'dB} &= n\cos^2\alpha + n\cos 2\alpha\tan^2\delta - m\sin\alpha\tan\delta,\\
\frac{d^2\Delta\alpha}{dB^2} &= -\tfrac{1}{2}\sin 2\alpha - \sin 2\alpha\tan^2\delta,\\
\frac{d^2\Delta\delta}{dA'^2} &= -mn\sin\alpha - n^2\sin^2\alpha\tan\delta,\\
\frac{d^2\Delta\delta}{dA'dB} &= -\frac{n}{2}\sin 2\alpha\tan\delta - m\cos\alpha,\\
\frac{d^2\Delta\delta}{dB^2} &= -\cos^2\alpha\tan\delta.
\end{aligned}\right\} \tag{21}$$

It will be sufficient to retain in $\Delta^2\alpha$ only the terms multiplied by $\tan^2\delta$, and in $\Delta^2\delta$ those multiplied by $\tan\delta$, and to put $A' = -0.34236 \sin \Omega = -\frac{v}{n}\sin\Omega$, and $B = -9''.2235\cos\Omega = -u\cos\Omega$, where Ω denotes the longitude of the moon's ascending node. Thus we get

$$\left.\begin{aligned}
\Delta^2\alpha &= \left[\frac{uv}{2}\cos 2\alpha\sin 2\Omega - \frac{u^2+v^2}{4}\sin 2\alpha\cos 2\Omega\right]\tan^2\delta,\\
\Delta^2\delta &= -\left[\frac{uv}{4}\sin 2\alpha\sin 2\Omega + \left(\frac{u^2-v^2}{8} + \frac{u^2+v^2}{8}\cos 2\alpha\right)\cos 2\Omega\right]\tan\delta.
\end{aligned}\right\} \tag{22}$$

Hence, if we put

$$\left.\begin{aligned}
a &= \tfrac{1}{15}(m + n\sin\alpha\tan\delta), & b &= \tfrac{1}{15}\cos\alpha\tan\delta,\\
a' &= n\cos\alpha, & b' &= -\sin\alpha,
\end{aligned}\right\} \tag{23}$$

the formulas for the whole effect of nutation will be

$$\left.\begin{aligned}
\Delta\alpha &= aA' + bB + E\\
&\quad + [0^s.0000103\cos 2\alpha\sin 2\Omega - 0^s.0000107\sin 2\alpha\cos 2\Omega]\tan^2\delta,\\
\Delta\delta &= a'A' + b'B\\
&\quad - [0''.000077\sin 2\alpha\sin 2\Omega + (0''.000023 + 0''.000080\cos 2\alpha)\cos 2\Omega]\tan\delta.
\end{aligned}\right\} \tag{24}$$

5. The effect of aberration is next to be considered. If α' and δ' denote the right ascension and declination of the star as affected by aberration, while α and δ denote the same unaffected by aberration, and $\frac{dX}{dt}$, $\frac{dY}{dt}$ and $\frac{dZ}{dt}$ denote the velocity of the earth projected on the three axes of coordi-

nates, and k denote the velocity of light, we have, R' being a fictitious distance to be eliminated,

$$\left.\begin{aligned} R' \cos\delta' \cos\alpha' &= \cos\delta\cos\alpha + \frac{1}{k}\frac{dX}{dt}, \\ R' \cos\delta' \sin\alpha' &= \cos\delta\sin\alpha + \frac{1}{k}\frac{dY}{dt}, \\ R' \sin\delta' &= \sin\delta \qquad\;\; + \frac{1}{k}\frac{dZ}{dt}. \end{aligned}\right\} \tag{25}$$

Whence are derived

$$\left.\begin{aligned} R'\cos\delta'\sin(\alpha'-\alpha) &= -\frac{1}{k}\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right), \\ R'\cos\delta'\cos(\alpha'-\alpha) &= \cos\delta + \frac{1}{k}\left(\frac{dX}{dt}\cos\alpha + \frac{dY}{dt}\sin\alpha\right), \\ R'\sin(\delta'-\delta) &= -\frac{1}{k}\left(\frac{dX}{dt}\sin\delta\cos\alpha + \frac{dY}{dt}\sin\delta\sin\alpha - \frac{dZ}{dt}\cos\delta\right) \\ &\quad - \frac{1}{2k^2}\tan\delta\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right)^2, \\ R'\cos(\delta'-\delta) &= 1 + \frac{1}{k}\left(\frac{dX}{dt}\cos\delta\cos\alpha + \frac{dY}{dt}\cos\delta\sin\alpha + \frac{dZ}{dt}\sin\delta\right) \\ &\quad + \frac{1}{2k^2}\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right)^2. \end{aligned}\right\} \tag{26}$$

From which, to quantities of the second order, we have

$$\left.\begin{aligned} \alpha'-\alpha &= -\frac{1}{k}\sec\delta\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right) \\ &\quad + \frac{1}{k^2}\sec^2\delta\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right)\left(\frac{dX}{dt}\cos\alpha + \frac{dY}{dt}\sin\alpha\right), \\ \delta'-\delta &= -\frac{1}{k}\left(\frac{dX}{dt}\sin\delta\cos\alpha + \frac{dY}{dt}\sin\delta\sin\alpha - \frac{dZ}{dt}\cos\delta\right) \\ &\quad - \frac{1}{2k^2}\tan\delta\left(\frac{dX}{dt}\sin\alpha - \frac{dY}{dt}\cos\alpha\right)^2 \\ &\quad + \frac{1}{k^2}\left(\frac{dX}{dt}\sin\delta\cos\alpha + \frac{dY}{dt}\sin\delta\sin\alpha - \frac{dZ}{dt}\cos\delta\right) \\ &\qquad \times\left(\frac{dX}{dt}\cos\delta\cos\alpha + \frac{dY}{dt}\cos\delta\sin\alpha + \frac{dZ}{dt}\sin\delta\right). \end{aligned}\right\} \tag{27}$$

If r is the radius vector of the earth, $\odot$ the sun's true longitude and ω the obliquity of the ecliptic,

$$X = -r\cos\odot, \quad Y = -r\sin\odot\cos\omega, \quad Z = -r\sin\odot\sin\omega. \tag{28}$$

And, if e denotes the eccentricity of the earth's orbit, Γ the longitude of the solar perigee and n the mean sidereal motion of the sun,

$$\left.\begin{aligned} \frac{dr}{dt} &= \frac{an}{\sqrt{1-e^2}}\,e\sin(\odot-\Gamma), \\ r\frac{d\odot}{dt} &= \frac{an}{\sqrt{1-e^2}}[1 + e\cos(\odot-\Gamma)]. \end{aligned}\right\} \tag{29}$$

Whence we derive

$$\left.\begin{aligned}\frac{dX}{dt} &= \frac{an}{\sqrt{1-e^2}}[\sin \odot + e \sin \Gamma],\\ \frac{dY}{dt} &= -\frac{an}{\sqrt{1-e^2}}\cos\omega\,[\cos \odot + e \cos \Gamma],\\ \frac{dZ}{dt} &= -\frac{an}{\sqrt{1-e^2}}\sin\omega\,[\cos \odot + e \cos \Gamma].\end{aligned}\right\} \tag{30}$$

By substituting these values in (27), making $\frac{an}{k\sqrt{1-e^2}} = \varkappa$, and omitting the terms which are independent of $\odot$, we have

$$\begin{aligned}\alpha' - \alpha = &-\varkappa \sec\delta\,[\sin\alpha \sin\odot + \cos\alpha\cos\omega\cos\odot]\\ &-\frac{\varkappa^2}{4}\sec^2\delta\,[(1+\cos^2\omega)\sin 2\alpha\cos 2\odot - 2\cos\omega\cos 2\alpha\sin 2\odot],\\ \delta' - \delta = &-\varkappa\,[\sin\delta\cos\alpha\sin\odot - (\cos\omega\sin\delta\sin\alpha - \sin\omega\cos\delta)\cos\odot]\\ &-\frac{\varkappa^2}{8}\tan\delta\,[((1+\cos^2\omega)\cos 2\alpha - \sin^2\omega)\cos 2\odot + 2\cos\omega\sin 2\alpha\sin 2\odot].\end{aligned}$$

In these formulas, terms multiplied by $\varkappa^2 e$ have been neglected, as also the terms in $\delta' - \delta$ multiplied by $\varkappa^2$ which are not also multiplied by $\tan\delta$. Substituting for $\varkappa$ Struve's value $20''.4451$, these formulas become

$$\left.\begin{aligned}\alpha' - \alpha = &-20''.4451\sec\delta\,[\sin\alpha\sin\odot + \cos\alpha\cos\omega\cos\odot]\\ &-0''.0009329\sec^2\delta\sin 2\alpha\cos 2\odot + 0''.0009295\sec^2\delta\cos 2\alpha\sin 2\odot,\\ \delta' - \delta = &-20''.4451\sin\delta\cos\alpha\sin\odot\\ &+20''.4451\cos\odot\,[\sin\delta\sin\alpha\cos\omega - \cos\delta\sin\omega]\\ &-0''.0004648\tan\delta\sin 2\alpha\sin 2\odot\\ &+[0''.0000402 - 0''.0004665\cos 2\alpha]\tan\delta\cos 2\odot.\end{aligned}\right\} \tag{31}$$

6. The values α, δ, $\odot$ and ω to be employed here are those affected by nutation. Hence, if we use values referred to the mean equinox of date, we must add to $\alpha' - \alpha$ the terms

$$\frac{d(\alpha'-\alpha)}{d\alpha}\Delta\alpha + \frac{d(\alpha'-\alpha)}{d\delta}\Delta\delta + \frac{d(\alpha'-\alpha)}{d\odot}\Delta\odot + \frac{d(\alpha'-\alpha)}{d\omega}\Delta\omega,$$

and to $\delta' - \delta$ the terms

$$\frac{d(\delta'-\delta)}{d\alpha}\Delta\alpha + \frac{d(\delta'-\delta)}{d\delta}\Delta\delta + \frac{d(\delta'-\delta)}{d\odot}\Delta\odot + \frac{d(\delta'-\delta)}{d\omega}\Delta\omega.$$

Those multiplied by $\Delta\odot$ and $\Delta\omega$ are of no importance, and it will be sufficient to put

$$\left.\begin{aligned}\Delta\alpha &= -[b\sin\alpha\sin\Omega + a\cos\alpha\cos\Omega]\tan\delta,\\ \Delta\delta &= -b\cos\alpha\sin\Omega + a\sin\alpha\cos\Omega,\end{aligned}\right\} \tag{32}$$

where $b = 6''.865$ and $a = 9''.2235$.

Then the terms to add to $\alpha' - \alpha$ are

$$\frac{20''.4451}{2} \tan\delta \sec\delta \left\{ \begin{array}{l} -(b + a\cos\omega)\sin 2\alpha \cos(\odot + \Omega) \\ +(b\cos\omega + a)\cos 2\alpha \sin(\odot + \Omega) \\ +(b - a\cos\omega)\sin 2\alpha \cos(\odot - \Omega) \\ -(b\cos\omega - a)\cos 2\alpha \sin(\odot - \Omega) \end{array} \right\}, \tag{33}$$

and to $\delta' - \delta$,

$$\frac{20''.4451}{4} \sin\delta \tan\delta \left\{ \begin{array}{l} -(b + a\cos\omega)\cos 2\alpha \cos(\odot + \Omega) \\ -(b\cos\omega + a)\sin 2\alpha \sin(\odot + \Omega) \\ +(b - a\cos\omega)\cos 2\alpha \cos(\odot - \Omega) \\ +(b\cos\omega - a)\sin 2\alpha \sin(\odot - \Omega) \\ +(b - a\cos\omega)\cos(\odot + \Omega) \\ -(b + a\cos\omega)\cos(\odot - \Omega) \end{array} \right\}. \tag{34}$$

Or, the numerical values of a, b and ω being substituted, we have

$$\alpha' - \alpha = \left\{ \begin{array}{l} -0^s.00005065 \sin 2\alpha \cos(\odot + \Omega) \\ \qquad\qquad +0^s.00005129 \cos 2\alpha \sin(\odot + \Omega) \\ -0^s.00000527 \sin 2\alpha \cos(\odot - \Omega) \\ \qquad\qquad +0^s.00000966 \cos 2\alpha \sin(\odot - \Omega) \end{array} \right\} \tan\delta \sec\delta, \tag{35}$$

$$\delta' - \delta = \left\{ \begin{array}{l} -0''.0003799 \cos 2\alpha \cos(\odot + \Omega) \\ \qquad\qquad -0''.0003847 \sin 2\alpha \sin(\odot + \Omega) \\ -0''.0000395 \cos 2\alpha \cos(\odot - \Omega) \\ \qquad\qquad -0''.0000725 \sin 2\alpha \sin(\odot - \Omega) \\ -0''.0000391 \cos(\odot + \Omega) \\ \qquad\qquad -0''.0003799 \cos(\odot - \Omega) \end{array} \right\} \sin\delta \tan\delta. \tag{36}$$

7. If we make

$$\left. \begin{array}{ll} C = -20''.4451 \cos\omega \cos\odot, & D = -20''.4451 \sin\odot, \\ c = \frac{1}{15}\cos\alpha \sec\delta, & d = \frac{1}{15}\sin\alpha \sec\delta, \\ c' = \tan\omega \cos\delta - \sin\alpha \sin\delta, & d' = \cos\alpha \sin\delta, \end{array} \right\} \tag{37}$$

we shall have the combined effect of nutation and aberration on the place of the star, terms of the second order being omitted, by the formulas

$$\left. \begin{array}{l} \alpha' - \alpha = aA' + bB + cC + dD + E, \\ \delta' - \delta = a'A' + b'B + c'C + d'D. \end{array} \right\} \tag{38}$$

If we wish to include the mean motion of the star from the beginning of the year, we must add, respectively, to these expressions the terms $(a + \mu)\tau$ and $(a' + \mu')\tau$, where for a, a' μ and μ' should be taken their values, not for date, but for the time $\frac{\tau}{2}$. Hence, if we make $A' + \tau = A$, the formulas become in this case,

$$\left. \begin{array}{l} \alpha' - \alpha = aA + bB + cC + dD + E + \mu\tau, \\ \delta' - \delta = a'A + b'B + c'C + d'D + \mu'\tau; \end{array} \right\} \tag{39}$$

and to our terms of the second order must be added in right ascension $-\frac{1}{2}\frac{da}{d\tau}\tau^2$ and in declination $-\frac{1}{2}\frac{da'}{d\tau}\tau^2$; or better, if μ and μ' in the last two equations denote their values at the beginning of the year, these terms will be $-\frac{1}{2}\frac{d^2\alpha}{d\tau^2}\tau^2$ and $-\frac{1}{2}\frac{d^2\delta}{d\tau^2}\tau^2$ where $\frac{d^2\alpha}{d\tau^2}$ and $\frac{d^2\delta}{d\tau^2}$ have the values (15). Neglecting the variation of m and n, and the terms not multiplied by $\tan\delta$ or $\tan^2\delta$, these terms of the second order are

$$\left.\begin{aligned} \Delta(\alpha'-\alpha) &= +0^s.000003\,\tau^2\sin\alpha\tan\delta - 0^s.000149\,\tau^2\cos\alpha\tan\delta \\ &\quad - 0^s.0000650\,\tau^2\sin 2\alpha\tan^2\delta, \\ \Delta(\delta'-\delta) &= +0''.000975\,\tau^2\sin^2\alpha\tan\delta. \end{aligned}\right\} \qquad (40)$$

8. In order that the subject of star reductions may be complete, it is necessary to consider the effect of orbital motion in double stars. The corrections of the right ascension and declination always have this form,

$$\left.\begin{aligned} \Delta\alpha &= a + bt + k\sin(u+K), \\ \Delta\delta &= a' + b't + k'\sin(u+K'), \end{aligned}\right\} \qquad (41)$$

where u is derived from the equation

$$u - e\sin u = n(t-T). \qquad (42)$$

All the quantities in these equations, except t and u, are constants to be derived from observation.

Construction of General Star Tables.

9. It will be convenient to divide the quantities A and B each into two parts, so that $A = A_\odot + A_\Omega$ and $B = B_\odot + B_\Omega$, where for the epoch 1870 the values of $A_\odot$, A_Ω, $B_\odot$, B_Ω (the numbers in brackets are logarithms) are

$$\left.\begin{aligned} A_\odot &= \tau + [6.5942]\sin\odot + [7.4644]\cos\odot - [8.4012]\sin 2\odot, \\ B_\odot &= -[7.9609]\sin\odot - [7.2370]\cos\odot - [9.7410]\cos 2\odot, \\ C &= -[1.27313]\cos\odot, \\ D &= -[1.31059]\sin\odot, \\ A_\Omega &= -[9.53457 + 0.4t]\sin\Omega + [7.6128]\sin 2\Omega, \\ B_\Omega &= -[0.96490]\cos\Omega + [8.9518]\cos 2\Omega, \\ E_\Omega &= -[7.4951 - 6.6t]\sin\Omega. \end{aligned}\right\} \qquad (43)$$

The term $E_\odot$ being neglected, we write

$$\left.\begin{aligned} \Delta_\odot\alpha &= aA_\odot + bB_\odot + cC + dD + \mu\tau, \\ \Delta_\odot\delta &= a'A_\odot + b'B_\odot + c'C + d'D + \mu'\tau, \\ \Delta_\Omega\alpha &= aA_\Omega + bB_\Omega + E_\Omega, \\ \Delta_\Omega\delta &= a'A_\Omega + b'B_\Omega. \end{aligned}\right\} \qquad (44)$$

To $\Delta_{\odot}\alpha$ and $\Delta_{\odot}\delta$ should be added the terms of the second order in aberration, and to $\Delta_{\Omega}\alpha$ and $\Delta_{\Omega}\delta$ the terms of the second order in nutation whenever they are sensible.

If we make

$$\left.\begin{aligned}
p_{\odot} &= -[1.31059]\, d + [6.5942]\, a - [7.9609]\, b\,, \\
q_{\odot} &= -[1.27313]\, c + [7.4644]\, a - [7.2370]\, b\,, \\
p_{2\odot} &= -[8.4012\ ]\, a + [5.7922] \sec^2\delta \cos 2\alpha\,, \\
q_{2\odot} &= -[9.7410\ ]\, b - [5.7938] \sec^2\delta \sin 2\alpha\,, \\
p'_{\odot} &= -[1.31059]\, d' + [6.5942]\, a' - [7.9609]\, b'\,, \\
q'_{\odot} &= -[1.27313]\, c' + [7.4644]\, a' - [7.2370]\, b'\,, \\
p'_{2\odot} &= -[8.4012\ ]\, a' - [6.6673] \tan\delta \sin 2\alpha\,, \\
q'_{2\odot} &= -[9.7410\ ]\, b' - [6.6688] \tan\delta \cos 2\alpha + [5.6042] \tan\delta\,,
\end{aligned}\right\} \tag{45}$$

we shall have, terms of the second order included,

$$\left.\begin{aligned}
\Delta_{\odot}\alpha &= p_{\odot} \sin\odot + q_{\odot}\cos\odot + p_{2\odot}\sin 2\odot + q_{2\odot}\cos 2\odot + \frac{d\alpha}{d\tau}\tau + \frac{1}{2}\frac{d^2\alpha}{d\tau^2}\tau^2, \\
\Delta_{\odot}\delta &= p'_{\odot}\sin\odot + q'_{\odot}\cos\odot + p'_{2\odot}\sin 2\odot + q'_{2\odot}\cos 2\odot + \frac{d\delta}{d\tau}\tau + \frac{1}{2}\frac{d^2\delta}{d\tau^2}\tau^2.
\end{aligned}\right\} \tag{46}$$

Let us make

$$\left.\begin{aligned}
p_{\odot} &= k_{\odot}\cos K_{\odot}\,, & p'_{\odot} &= k'_{\odot}\cos K'_{\odot}\,, \\
q_{\odot} &= k_{\odot}\sin K_{\odot}\,, & q'_{\odot} &= k'_{\odot}\sin K'_{\odot}\,, \\
p_{2\odot} &= k_{2\odot}\cos K_{2\odot}\,, & p'_{2\odot} &= k'_{2\odot}\cos K'_{2\odot}\,, \\
q_{2\odot} &= k_{2\odot}\sin K_{2\odot}\,, & q'_{2\odot} &= k'_{2\odot}\sin K'_{2\odot}\,.
\end{aligned}\right\} \tag{47}$$

Then equations (46) take the form

$$\left.\begin{aligned}
\Delta_{\odot}\alpha &= \frac{d\alpha}{d\tau}\tau + \frac{1}{2}\frac{d^2\alpha}{d\tau^2}\tau^2 + k_{\odot}\sin(\odot + K_{\odot}) + k_{2\odot}\sin(2\odot + K_{2\odot})\,, \\
\Delta_{\odot}\delta &= \frac{d\delta}{d\tau}\tau + \frac{1}{2}\frac{d^2\delta}{d\tau^2}\tau^2 + k'_{\odot}\sin(\odot + K'_{\odot}) + k'_{2\odot}\sin(2\odot + K_{2\odot})\,.
\end{aligned}\right\} \tag{48}$$

10. To compute the variations of $\Delta_{\odot}\alpha$ and $\Delta_{\odot}\delta$ for a certain interval of time as 10 years, we compute the variations of $p_{\odot}$, $q_{\odot}$, etc., in that interval; calling them $\delta p_{\odot}$, $\delta q_{\odot}$, etc., and certain very small terms being neglected, we have evidently these equations:

$$\left.\begin{aligned}
\delta\,.\,\Delta_{\odot}\alpha &= \delta p_{\odot}\sin\odot + \delta q_{\odot}\cos\odot + \delta p_{2\odot}\sin 2\odot + \delta q_{2\odot}\cos 2\odot \\
&\qquad + 10\frac{d^2\alpha}{d\tau^2}\tau + k_{\odot}\cos(\odot + K_{\odot})\,\delta\odot\,, \\
\delta\,.\,\Delta_{\odot}\delta &= \delta p'_{\odot}\sin\odot + \delta q'_{\odot}\cos\odot + \delta p'_{2\odot}\sin 2\odot + \delta q'_{2\odot}\cos 2\odot \\
&\qquad + 10\frac{d^2\delta}{d\tau^2}\tau + k'_{\odot}\cos(\odot + K'_{\odot})\,\delta\odot\,.
\end{aligned}\right\} \tag{49}$$

The value of $\delta\odot$ is [6.0057] sin $(\odot - 15°)$; substituting this, we have

$$\left.\begin{aligned}
\delta\,.\,\Delta_{\odot}\alpha &= 10\frac{d^2\alpha}{d\tau}\tau - [5.7047]\,k_{\odot}\sin(K_{\odot}+15°) + \delta p_{\odot}\sin\odot + \delta q_{\odot}\cos\odot \\
&\quad + [\delta p_{2\odot} + [5.7047]\,k_{\odot}\cos(K_{\odot}-15°)]\sin 2\odot \\
&\quad + [\delta q_{2\odot} + [5.7047]\,k_{\odot}\sin(K_{\odot}-15°)]\cos 2\odot\,, \\
\delta\,.\,\Delta_{\odot}\delta &= 10\frac{d^2\delta}{d\tau^2}\tau - [5.7047]\,k'_{\odot}\sin(K'_{\odot}+15°) + \delta p'_{\odot}\sin\odot + \delta q'_{\odot}\cos\odot \\
&\quad + [\delta p'_{2\odot} + [5.7047]\,k'_{\odot}\cos(K'_{\odot}-15°)]\sin 2\odot \\
&\quad + [\delta q'_{2\odot} + [5.7047]\,k'_{\odot}\sin(K'_{\odot}-15°)]\cos 2\odot\,.
\end{aligned}\right\} \tag{50}$$

As in the case of $\Delta_{\odot}\alpha$ and $\Delta_{\odot}\delta$, these quantities can be made to take the form

$$\left.\begin{aligned}
\delta\,.\,\Delta_{\odot}\alpha &= a + b\tau + h_{\odot}\sin(\odot + H_{\odot}) + h_{2\odot}\sin(2\odot + H_{2\odot})\,, \\
\delta\,.\,\Delta_{\odot}\delta &= a' + b'\tau + h'_{\odot}\sin(\odot + H'_{\odot}) + h'_{2\odot}\sin(2\odot + H'_{2\odot})\,.
\end{aligned}\right\} \tag{51}$$

Except for stars near either pole, the first and last terms of these equations may be neglected, and regard be had in computing $\delta p_{\odot}$, $\delta q_{\odot}$, etc., only to the variations of c, d, c' and d' in the formulas for $p_{\odot}$, $q_{\odot}$, etc. Then

$$\left.\begin{aligned}
\delta\,.\,\Delta_{\odot}\alpha &= 10\frac{d^2\alpha}{d\tau^2}\tau + h_{\odot}\sin(\odot + H_{\odot})\,, \\
\delta\,.\,\Delta_{\odot}\delta &= 10\frac{d^2\delta}{d\tau^2}\tau + h'_{\odot}\sin(\odot + H'_{\odot})\,.
\end{aligned}\right\} \tag{52}$$

11. In computing $\Delta_{\odot}\alpha$ and $\Delta_{\odot}\delta$, we may either suppose $k_{\odot}$, $K_{\odot}$, $k'_{\odot}$ and $K'_{\odot}$ constant throughout the year, and afterwards add to $\Delta_{\odot}\alpha$ and $\Delta_{\odot}\delta$ thus obtained, the proper fractional part of $h_{\odot}$ sin $(\odot + H_{\odot})$ and $h'_{\odot}$ sin $(\odot + H'_{\odot})$ for the fraction of the year; or, we may make them vary from date to date. For a star, whose declination is within the limits $\pm 65°$, there is, however, no need to attend to this correction.

Having formed a table of $\odot$ for every 10 sidereal days, beginning with the fictitious year, we can readily get $\odot$ for the time of the star's transit over the fictitious meridian with the constant interpolation factor $\dfrac{\alpha - 18^h40^m}{240^h}$, and thus form the arguments $\odot + K_{\odot}$, $2\odot + K_{2\odot}$, $\odot + H_{\odot}$, etc. Terms with small coefficients can be most readily formed by means of a Traverse Table.

12. We can reduce $\Delta_{\Omega}\alpha$ and $\Delta_{\Omega}\delta$ to the forms, terms of the second order included,

$$\left.\begin{aligned}
\Delta_{\Omega}\alpha &= k_{\Omega}\sin(\Omega + K_{\Omega}) + k_{2\Omega}\sin(2\Omega + K_{2\Omega})\,, \\
\Delta_{\Omega}\delta &= k'_{\Omega}\sin(\Omega + K'_{\Omega}) + k'_{2\Omega}\sin(2\Omega + K'_{2\Omega})\,.
\end{aligned}\right\} \tag{53}$$

by making

$$\left.\begin{aligned}
k_{\Omega} \cos K_{\Omega} &= -[9.53457 + 0.4t]\, a - 0^s.0031, \\
k_{\Omega} \sin K_{\Omega} &= -[0.96490]\, b, \\
k_{2\Omega} \cos K_{2\Omega} &= [7.6128\ ]\, a + [5.0114] \cos 2\alpha \tan^2 \delta, \\
k_{2\Omega} \sin K_{2\Omega} &= [8.9518\ ]\, b - [5.0294] \sin 2\alpha \tan^2 \delta, \\
k'_{\Omega} \cos K'_{\Omega} &= -[9.53457 + 0.4t]\, a', \\
k'_{\Omega} \sin K'_{\Omega} &= -[0.96490]\, b', \\
k'_{2\Omega} \cos K'_{2\Omega} &= [7.6128\ ]\, a' - [5.8865] \sin 2\alpha \tan \delta, \\
k'_{2\Omega} \sin K'_{2\Omega} &= [8.9518\ ]\, b' - [5.9031] \cos 2\alpha \tan \delta - [5.3617] \tan \delta .
\end{aligned}\right\} \tag{54}$$

But perhaps it will be as well to adopt the formulas

$$\left.\begin{aligned}
\Delta_{\Omega}\alpha &= aA_{\Omega} + bB_{\Omega} + E_{\Omega}, \\
\Delta_{\Omega}\delta &= a'A_{\Omega} + b'B_{\odot},
\end{aligned}\right\} \tag{55}$$

or,

$$\left.\begin{aligned}
\Delta_{\Omega}\alpha &= f_{\Omega} + g_{\Omega} \sin (G_{\Omega} + \alpha) \tan \delta, \\
\Delta_{\Omega}\delta &= \phantom{f_{\Omega} +}\ g_{\Omega} \cos (G_{\Omega} + \alpha).
\end{aligned}\right\} \tag{56}$$

13. For stars near either pole, it will be well to construct tables giving, with the arguments $\odot + \Omega$ and $\odot - \Omega$, the values of the small terms in (33) and (34). These will be most readily computed with the aid of a Traverse Table, when they have been reduced to the forms

$$\left.\begin{aligned}
\Delta_{\odot+\Omega}\alpha &= k_{\odot+\Omega} \sin (\odot + \Omega + K_{\odot+\Omega}), \\
\Delta_{\odot+\Omega}\delta &= k'_{\odot+\Omega} \sin (\odot + \Omega + K'_{\odot+\Omega}), \\
\Delta_{\odot-\Omega}\alpha &= k_{\odot-\Omega} \sin (\odot - \Omega + K_{\odot-\Omega}), \\
\Delta_{\odot-\Omega}\delta &= k'_{\odot-\Omega} \sin (\odot - \Omega + K'_{\odot-\Omega}).
\end{aligned}\right\} \tag{57}$$

14. Tables for $\Delta_{☾}\alpha$ and $\Delta_{☾}\delta$ may be computed in the same way. For, by making

$$\left.\begin{aligned}
k_{☾} \cos K_{☾} &= -[7.6075]\, a, \\
k_{☾} \sin K_{☾} &= -[8.9474]\, b, \\
k'_{☾} \cos K'_{☾} &= -[7.6075]\, a', \\
k'_{☾} \sin K'_{☾} &= -[8.9474]\, b',
\end{aligned}\right\} \tag{58}$$

these quantities take the form

$$\left.\begin{aligned}
\Delta_{☾}\alpha &= k_{☾} \sin (2☾ + K_{☾}), \\
\Delta_{☾}\delta &= k'_{☾} \sin (2☾ + K'_{☾}).
\end{aligned}\right\} \tag{59}$$

In tabulating these quantities it will be better to make sidereal time the argument rather than ☾; and if they are to be tabulated for several stars, they should be interpolated forwards a time equal to $\alpha - 18^h 40^m$, so that the argument may be the same for the transits of all the stars on the same sidereal day. If these quantities are tabulated for every tenth of a sidereal day throughout the period of the argument, we may take advantage of the fact that this period is almost exactly 13.7 sidereal days, to arrange the table so that there will be an interval of a sidereal day between successive values of the argument.

On the Places and Proper Motions of β^1 Scorpii, β and η Draconis and 61^1 Cygni.

In this discussion almost all the material to be derived from the well known collections of reduced observations have been employed. To accord with Dr. Gould's *Standard Places of Fundamental Stars*, the right ascensions as given in these collections have been reduced to the equinox of Argelander's DLX *Stellarum Positiones Mediæ.* Also the systematic corrections given by Dr. Auwers in the *Astronomische Nachrichten*, Vol. LXV, 370, 377-382, have been applied to the declinations.

β^1 Scorpii.

Adopting for a provisional mean place of this star, that of the *Greenwich Seven Year Catalogue for* 1860, we have for any time, with a proper motion zero in both coordinates,

$$\alpha = 15^h57^m18^s.08+3^s.47735\,(t-1860)+0^s.0000711\,(t-1860)^2,$$
$$\delta = -19°25'8''.37-10''.2316\,(t-1860)+0''.002203\,(t-1860)^2.$$

Constructing an ephemeris from these formulas and comparing it with the observations, both as published and corrected, we obtain the differences in columns I and II of the following table. From those in column II are derived, by the method of least squares, the following normal equations, in which $\Delta\alpha$ and $\Delta\delta$ denote the corrections of the provisional mean right ascension and declination, and μ and μ' are the proper motions in those coordinates for 1860 :

$$\begin{aligned}
37.5\,\Delta\alpha - 810\mu &= +1^s.0865\\
-810\,\Delta\alpha + 40379\mu &= -64^s.793\\
34\,\Delta\delta - 746\mu' &= +38''.685\\
-746\,\Delta\delta + 39090\mu' &= -1279''.485
\end{aligned}$$

whence

$$\Delta\alpha = -0^s.010,\quad \Delta\delta = +0''.72,\quad \mu = -0^s.00180,\quad \mu' = -0''.0189;$$

and the corrected formulas for the mean place of the star are

$$\alpha = 15^h57^m\,18^s.070 + 3^s.47555\,(t-1860) + 0^s.0000712\,(t-1860)^2,$$
$$\delta = -19°25\;7''.65\quad -10''.2505\,(t-1860) + 0'\,.002200\,(t-1860)^2.$$

The differences of these and the corrected observations constitute column III of the table.

Authority.	Right Ascension. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.	Declination. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.
				s	s	s						
Bradley (*Fund. Astr.*)	1755	10	2	+ .118	+ .118	− .062	1755	8	2	+ 2″.36	+ 2″.36	− 0″.31
Piazzi	1800	44	2	+ .145	+ .206	+ .108	1800	28	2	+ 4.54	+ 2.51	+ 0.67
Airy, *Camb. Cat*	1829	12	2	+ .037	+ .037	− .009	1829	22	2	+ 2.17	+ 0.80	− 0.50
Pond	1830	13	2	+ .137	+ .137	+ .093	1830	12	2	+ 4.04	+ 1.76	+ 0.48
Struve, *Cat. Gen*	1830	5	2	+ .107	+ .067	+ .023	1830	5	2	+ 1.74	+ 0.76	− 0.52
Johnson, *St. Helena*	1830	9	2	+ .047	+ .907	− .037	1830	14	2	+ 1.74	+ 1.94	+ 0.66
Robinson, *Arm. Cat*	1834	4	¼	− .081	− .051	− .088	1853	2	¼	+ 1.57	+ 1.27	+ 0.42
Taylor, *Madras*	1837	14	1	+ .090	+ .010	− .020	1837	3	¼	+ 1.02	+ 0.66	− 0.49
Greenwich, 12-*Year Cat*	1838	51	3	− .061	+ .022	− .001	1838	37	3	+ 0.49	+ 1.07	0.00
Greenwich, 12-*Year Cat*	1845	57		+ .925			1844	41		1.01		
Gilliss, *Washington*	1840	30	1	+ .020	+ .020	− .006						
Edinburgh	1844	34	3	+ .014	+ .014	− .005	1842	15	3	+ 1.40	+ 0.22	− 0.84
Radcliffe Cat	1848	18	3	+ .055	− .025	− .037	1850	5	1	+ 1.39	+ 2.05	+ 1.14
Greenwich	1851	54	4	+ .007	− .016	− .022	1851	47	4	+ 0.90	+ 1.02	+ 0.13
Brussels							1856	24	3	+ 0.64	+ 0.64	− 0.15
Greenwich	1857	14	4	+ .005	− .008	− .003	1857	32	4	+ 0.06	+ 0.45	− 0.32
Greenwich	1862	12	2	− .008	− .021	− .007	1862	8	2	+ 0.11	+ 0.50	− 0.18
Paris	1860	115	4	− .005	+ .011	+ .022	1860	93	4	+ 0.46	+ 0.57	+ 0.15

η *Draconis.*

Deriving a provisional place from the same authority as for the previous star, with no proper motion, we obtain the formulas:

$$\alpha = 16^h 22^m 6^s.23 + 0^s.79943\,(t-1860) + 0^s.0000943\,(t-1860)^2,$$
$$\delta = +\,61°49'54''.16 - 8''.3086\,(t-1860) + 0''.000548\,(t-1860)^2.$$

Comparing these with the published and corrected observations, we have columns I and II of the following table, and we find from the latter the normal equations

$$\begin{aligned} 36\,\varDelta\alpha - 647\mu &= -\ \ 4^s.983 \\ -\,647\,\varDelta\alpha + 21146\mu &= +\ 112.538 \\ 39\,\varDelta\delta - 766\mu' &= -\ \ 35''.23 \\ -\,766\,\varDelta\delta + 38618\mu' &= +\ 2293''.37 \end{aligned}$$

whence

$$\varDelta\alpha = -\,0^s.095, \quad \varDelta\delta = +\,0''.43, \quad \mu = +\,0^s.00242, \quad \mu' = +\,0''.0679;$$

and the corrected formulas for the mean place of the star are

$$\alpha = 16^h 22^m 6^s.135 + 0^s.80185\,(t-1860) + 0^s.0000924\,(t-1860)^2,$$
$$\delta = +\,61°49'54''.59 - 8''.2407\,(t-1860) + 0''.000551\,(t-1850)^2.$$

10

The differences of these and the corrected observations constitute column III of the table.

Authority.	Right Ascension.						Declination.					
	Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	Obs.—Cal. II.	Obs.—Cal. III.	Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	Obs.—Cal. II.	Obs.—Cal. III.
Bradley (*Fund. Astr.*)	1755			s	s	s	1755	5	2	$-7''.41$	$-7''.41$	$+0''.74$
Piazzi	1800	56	2	−.804	−.417	−.170	1800	9	1	− 3.65	− 2.85	+ 0.79
Groombridge	1810	15	2	−.295	−.049	+.172	1810	81	2	− 2.96	− 2.51	+ 0.45
Struve, *Cat. Gen*	1830	8	3	−.022	−.062	+.108	1830	8	3	− 1.41	− 1.47	+ 0.14
Pond	1830	10	2	−.182	−.182	−.012	1830	10	2	− 1.01	− 0.94	+ 0.67
Robinson, *Arm. Cat*	1833	2	½	−.409	−.469	−.306	1832	7	2	− 0.86	− 0.82	− 0.71
Taylor, *Madras*	1835	3	¼	−.123	−.203	−.046	1835	5	1	− 2.03	− 1.49	− 0.22
Greenwich, 12-*Year Cats*	1841	80	4	−.059	−.118	+.024	1841	269	4	− 0.38	− 0.60	+ 0.26
Gillis, *Washington*	1841	22	2	−.488	−.188	−.046						
Henderson	1841	5	1	−.045	−.045	+.097	1842	19	2	− 0.49	− 0.76	+ 0.03
Radcliffe	1842	18	3	−.070	−.150	−.010	1848	6	2	− 0.45	− 0.05	+ 0.34
Washington	1853	13	1	−.108	−.108	+.095	1850	7	1	+ 0.51	+ 0.34	+ 0.59
Greenwich	1851	43	4	−.080	−.123	−.005	1851	71	4	+ 0.26	− 0.13	+ 0.05
Brussels	1856	37	2	−.074	−.074	+.031	1855	14	2	− 0.02	− 0.02	− 0.11
Paris	1857	14	2	−.223	−.207	−.105	1858	54	4	+ 0.75	+ 0.32	+ 0.03
Greenwich	1858	39	4	−.044	−.057	+.043	1858	41	4	− 0.15	− 0.27	− 0.56
Radcliffe	1859	8	2	−.043	−.123	−.025	1860	6	2	+ 0.41	− 0.30	− 0.13
Greenwich	1863	7	1	−.173	−.186	−.098	1863	7	1	+ 0.44	+ 0.32	− 0.32

β *Draconis.*

The provisional place of this star derived from the same source, with no proper motion, is

$$\alpha = 17^h 27^m 16^s.32 + 1^s 35307\,(t-1860) + 0^s.0000256\,(t-1860)^2,$$
$$\delta = +\,52^\circ 24' 22''.71 - 2''.8543\,(t-1860) + 0''.000983\,(t-1860)^2.$$

Comparison of this with the published and corrected observations are given in columns I and II of the following table. From column II we find the normal equations

$$\begin{aligned} 35.6\Delta\alpha - 744\mu &= -1^s.635 \\ -744\ \Delta\alpha + 39800\mu &= -\ 32^s.231 \\ 43\ \Delta\delta - 944\mu' &= +\ 15''.34 \\ -944\ \Delta\delta + 45848\mu' &= -440''.33 \end{aligned}$$

whence

$$\Delta\alpha = -0^s.103\,,\quad \Delta\delta = +0''.27\,,\quad \mu = -0\,.00273\,,\quad \mu' = -0''.0041\,;$$

and the corrected formulas for the mean place of the star become,

$$\alpha = 17^h 27^m 16^s.217 + 1^s.35034\,(t-1860) + 0^s.0000257\,(t-1860)^2,$$
$$\delta = +\,52^\circ 24' 22''.98 - 2''.8584\,(t-1860) + 0''.000979\,(t-1860)^2.$$

Comparisons of these, with the corrected observations, constitute column III of the table.

Authority.	Right Ascension. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.	Declination. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.
				s	s	s						
Bradley (*Fund. Astr.*)	1755	10	2	+.197	+.197	+.012	1755	39	2	+ 0″.05	+ 0″.05	− 0″.61
Piazzi	1800	44	2	−.188	+.121	+.060	1800	18	2	+ 1.10	+ 1.40	+ 0.51
Groombridge	1810	11	1	−.220	−.067	−.101	1810	163	2	+ 0.61	+ 1.01	+ 0.55
Pond	1830	26	2	+.039	+.039	+.060	1830	133	3	+ 0.57	+ 0.31	− 0.08
Argelander	1830	101	4	−.051	−.051	−.030	1830	103	4	+ 0.57	+ 0.44	+ 0.05
Taylor	1835	3	½	+.201	+.121	+.156	1835	5	1	+ 0.70	+ 1.21	+ 0.84
Henderson							1839	11	2	+ 0.41	+ 0.05	− 0.30
Greenwich, 12-*Year Cats*	1842	79	4	−0.75	−.114	−.061	1840	143	4	+ 0.30	+ 0.18	− 0.17
Robinson, *Arm. Cat*	1842	1	½	−.469	−.049	−.446	1838	11	2	+ 0.87	− 0.06	− 0.42
Radcliffe, *Cat*	1844	9	2	−.030	−.110	−.051	1844	7	2	+ 0.46	+ 0.44	+ 0.11
Washington	1847	24	2	−.121	−.121	−.053	1847	34	2	+ 0.36	+ 0.36	+ 0.04
Greenwich	1851	29	3	−.032	−.045	+.034	1851	37	4	+ 0.43	+ 0.06	− 0.25
Brussels	1856	60	2	−.032	−.032	+.060	1857	11	1	+ 0.48	+ 0.48	+ 0.20
Radcliffe	1857	12	2	−.135	−.135	−.040	1857	15	2	+ 1.15	+ 1.00	+ 0.72
Greenwich	1859	45	4	+.004	−.009	+.091	1859	44	4	0.00	+ 0.02	− 0.25
Paris	1860	17	3	−.143	−.127	−.024	1858	44	4	+ 0.57	+ 0.36	+ 0.08
Greenwich	1861	7	2	−.123	−.136	−.030	1861	7	2	− 0.03	− 0.01	− 0.28

61^1 *Cygni.*

From Dr. Auwers' elements of the position of this star in the *Astronomische Nachrichten*, Vol. LIX, p. 354, we have

$$\alpha = \quad 21^h 0^m 37^s.396 + 2^s.67878\,(t-1860) + 0^s.0000207\,(t-1860)^2,$$
$$\delta = +\,38°3'46''.86 \;+ 17''.4510\,(t-1860) + 0''.001493\,(t-1860)^2,$$

including the proper motion in R. A., $\mu = +\,0^s.34512$, and in Dec., $\mu' = +\,3''.2311$.

On account of the importance of Bradley's two transit observations in determining the proper motion of this star, they have been reduced anew. With the equatorial intervals of the wires as given in the Preface to the Observations, and the clock corrections and constants of the instrumental corrections given by Bessel in the *Fundamenta Astronomiae*, the resulting mean right ascensions differ more than $0^s.5$. If, however, for 1753, October 8, we derive the clock correction from the observations of α Bootis, α Aquilae, α Cygni and α Piscis Australis, using Dr. Gould's values of their right ascensions, we have for October 8, $18^h 5^m$, $+\,23^s.13$, instead of Bessel's value $+\,23^s.58$, or a correction of Bessel's clock correction for that night of $-\,0^s.43$. Making this correction, we have for the apparent R. A. for the time of transit at Greenwich,

			Wires
1753, Sept. 25		$20^h 55^m 54^s.54$	1
	Oct. 8	$20^h 55^m 54^s.38$	3

Reducing these to mean place at date, the effect of the two terms depending on ☾ being added, and taking the annual variation in mean R. A. at

the epoch 1754 to be + $2^s.6695$, which is sufficiently accurate, we have, as the mean R. A. for 1755.0,

$$\text{Sept. } 25 \quad 20^h 55^m 56^s.23$$
$$\text{Oct. } 8 \quad 20^h 55^m 56^s.32$$

and, combined by giving double weight to the latter,

$$20^h 55^m 56^s.29$$

as the mean result from Bradley's observations.

Comparing the provisional formulas, with the different authorities and the corrected values as for the other stars, we have columns I and II of the following table. For the declinations, the normal equations resulting from column II are

$$54\, \Delta\delta - 1092\, \Delta\mu' = - 4''.59$$
$$-1092\, \Delta\delta + 48478\, \Delta\mu' = + 381''.45$$

from which

$$\Delta\delta = + 0''.14, \quad \Delta\mu' = + 0''.0109.$$

With regard to the right ascensions, it has been found more convenient to plot the residuals and draw a parabolic curve so as most nearly to represent them. Thus the correction to Auwers' formula is found to be

$$+ 0^s.041 + 0^s.00653\,(t - 1860) + 0^s.000054\,(t - 1860)^2.$$

Then the formulas for the corrected place are

$$\alpha = \quad 21^h 0^m 37^s.437 + 2^s.86531\,(t - 1860) + 0^s.0000747\,(t - 1860)^2,$$
$$\delta = + 38^\circ 3' 47''.00 + 17''.4619\,(t - 1860) + 0''.001493\,(t - 1860)^2,$$

in which the right ascension includes the term $+ 0^s.0000536\,(t - 1860)^2$ due to variability of proper motion.

Column III of the table shows how nearly these represent the several authorities.

Authority.	Right Ascension. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.	Declination. Mean Year.	No. Obs.	Weight.	Obs.—Cal. I.	II.	III.
				s	s	s						
Bradley (*corrected*)	1755	3	1	− .062	− .062	− .013	1754	4	2	− 1″.27	− 1″.15	− 0″.12
Piazzi	1805	28	2	− .384	− .169	− .014	1805	21	2	+ 2.32	+ 0.29	+ 0.76
Bessel	1816	85	2	− .131	− .131	+ .011	1816	20	2	− 0.29	− 0.79	− 0.45
Struve	1824	4	2	− .021	− .061	+ .063	1824	4	2	+ 0.42	+ 0.12	+ 0.38
Argelander	1830	62	4	− .091	− .091	+ .015	1830	32	4	+ 0.12	− 0.22	− 0.03
Pond	1830	80	0	+ .349			1830	242	4	+ 0.92	+ 0.10	+ 0.29
Taylor	1837	5	0	+ .381			1835	7	2	− 1.06	− 0.52	− 0.37
Greenwich	1839	55	4	− .089	− .098	− .026	1838	57	4	− 0.73	− 0.67	− 0.57
Henderson	1841	27	3	− .053	− .053	+ .011	1841	19	3	0.00	− 0.37	− 0.29
Greenwich	1834	50	4	− .059	− .068	− .018	1844	53	4	− 0.13	− 0.19	− 0.15
Washington	1847	152	2	− .020	− .020	− .015	1847	102	3	− 0.06	− 0.22	− 0.21
Radcliffe	1850	28	3	+ .071	− .009	+ .010	1853	20	3	+ 1.47	+ 0.94	+ 0.88
Greenwich	1851	62	4	+ .010	− .013	000	1852	52	4	+ 0.18	+ 0.14	+ 0.09
Brussels	1856	78	3	− .052	− .052	− .068	1857	26	3	+ 0.32	+ 0.32	+ 0.22
Greenwich	1857	51	41	+ .034	+ .021	− .001	1857	53	4	− 0.36	+ 0.24	+ 0.14
Radcliffe	1858	13	2	+ .030	+ .030	+ .001	1858	9	2	+ 0.57	+ 0.20	+ 0.09
Paris	1860	91	4	+ .017	+ .033	+ .008	1860	62	4	+ 0.01	+ 0.02	− 0.12
Greenwich	1863	14	2	+ .082	+ .069	+ .008	1863	16	2	− 0.59	− 0.29	− 0.46

MEMOIR No. 8.

Determination of the Elements of a Circular Orbit.

(Proceedings of the American Academy of Arts and Sciences, Vol. VIII, pp. 201–209, 1870.)

The following problem seems to possess some interest, and I have not, in my reading, met with any discussion of it:

To determine the elements of the orbit of a planet or satellite, which moves in a circle in the plane of the ecliptic, from three observations of its direction from the earth, made at equal intervals of time; the positions of the earth and the central body at these times being known, but the sum of the masses of the central body and the planet or satellite being unknown.

Or, geometrically stated—

In a plane, given a point as center and three straight lines, required to describe a circle, so that the arcs intercepted between the first and second, and the second and third, lines may be equal.

Let generally R denote the distance of the central body from the earth;
" " L its longitude as seen from the earth;
" " r the radius of the orbit of the planet;
" " λ its longitude as seen from the earth;
" " χ its longitude as seen from the central body.

Moreover, employ the subscripts $(_{-1})$, $(_0)$, $(_1)$ to denote the special values of the above quantities, which have place respectively at the three times of observation in their order.

If a perpendicular be let fall from the central body on the straight line which joins the earth and the body whose orbit is to be determined, its length is obviously

$$R \sin (\lambda - L);$$

another expression for the length of the same line is

$$r \sin (\chi - \lambda).$$

Hence, for the three times of observation, the three equations

$$\begin{aligned} r \sin (\chi_{-1} - \lambda_{-1}) &= R_{-1} \sin (\lambda_{-1} - L_{-1}), \\ r \sin (\chi_0 - \lambda_0) &= R_0 \sin (\lambda_0 - L_0), \\ r \sin (\chi_1 - \lambda_1) &= R_1 \sin (\lambda_1 - L_1). \end{aligned}$$

But, since the orbit is circular, χ increases uniformly with the time, and, consequently, $\chi_0 - \chi_{-1} = \chi_1 - \chi_0 = \eta$ suppose.

Thus the above equations may be written

$$\begin{aligned} r \sin (\chi_0 - \eta - \lambda_{-1}) &= R_{-1} \sin (\lambda_{-1} - L_{-1}) = a_{-1}, \\ r \sin (\chi_0 \quad - \lambda_0) &= R_0 \sin (\lambda_0 - L_0) \quad = a_0, \\ r \sin (\chi_0 + \eta - \lambda_1) &= R_1 \sin (\lambda_1 - L_1) \quad = a_1, \end{aligned}$$

which serve to determine the three unknown quantities r, χ_0 and η; and it will be noticed that their right hand members are known quantities.

If the sum of the masses of the central body and the body whose orbit is sought is denoted by μ, and the common interval of time between the observations by t,

$$\eta = t\sqrt{\frac{\mu}{r^3}};$$

thus, if μ were known, two observations would suffice to determine the orbit; but if μ is not known, η must be regarded as an independent unknown. Hence, the necessity for the restriction put at the end of the statement of the problem. Also by this restriction the problem is made to depend on the solution of an algebraical equation instead of a transcendental one.

The equations can be simplified by taking two unknown quantities ω and σ, instead of χ_0 and η, such that

$$\omega = \chi_0 - \frac{\lambda_1 + \lambda_{-1}}{2},$$

$$\sigma = \eta - \frac{\lambda_1 - \lambda_{-1}}{2},$$

and putting

$$\delta = \frac{\lambda_1 + \lambda_{-1}}{2} - \lambda_0,$$

then the equations become

$$\begin{aligned} r \sin (\omega - \sigma) &= a_{-1}, \\ r \sin (\omega + \delta) &= a_0, \\ r \sin (\omega + \sigma) &= a_1, \end{aligned}$$

or,

$$\begin{aligned} r \sin \omega \cos \sigma &= \frac{a_1 + a_{-1}}{2}, \\ r \sin (\omega + \delta) &= a_0, \\ r \cos \omega \sin \sigma &= \frac{a_1 - a_{-1}}{2}. \end{aligned}$$

If $r \sin \omega$ and $r \cos \omega$ are eliminated from these equations, and we make

$$\frac{a_1 + a_{-1}}{2a_0} \cos \delta = \mathrm{a} = c \cos \beta,$$

$$\frac{a_1 - a_{-1}}{2a_0} \sin \delta = \mathrm{b} = c \sin \beta,$$

where c may be taken as positive and the quadrant of β becomes determinate, or β may be assumed between the limits $\pm 90°$, there will be obtained, for the determination of σ, the equation

$$\sin 2\sigma = 2c \sin (\sigma + \beta) .$$

The computation of c and β may be facilitated by introducing the auxiliary quantities k and ζ, such that

$$k \sin \zeta = \frac{a_{-1}}{\sqrt{2a_0}}, \quad k \cos \zeta = \frac{a_1}{\sqrt{2a_0}},$$

then

$$c \cos \beta = k \cos (45° - \zeta) \cos \delta, \quad c \sin \beta = k \sin (45° - \zeta) \sin \delta .$$

It is evident that the determination of σ depends on the solution of an equation of the fourth degree; but its value can be very readily obtained from the above equation by the tentative process; and then r and ω by the equations

$$r \sin \omega = \frac{a_0 k \cos (45° - \zeta)}{\cos \sigma}, \quad r \cos \omega = \frac{a_0 k \sin (45° - \zeta)}{\sin \sigma},$$

and finally χ_0 and η by means of the relations given above.

There is a very simple geometrical construction of the roots of the equation in σ. Making $\cos \sigma = x$, and $\sin \sigma = y$, the values of x and y are the coordinates of the intersections of the curves whose equations are

$$x^2 + y^2 = 1, \quad (x - \mathrm{a})(y - \mathrm{b}) = \mathrm{ab} .$$

Consequently, if we construct the equilateral hyperbola whose equation is $xy = \pm 1$, and from a point on it, whose coordinates are

$$x' = -\frac{\mathrm{a}}{\sqrt{\pm \mathrm{ab}}}, \quad y' = -\frac{\mathrm{b}}{\sqrt{\pm \mathrm{ab}}},$$

as center, we describe a circle whose radius is $\dfrac{1}{\sqrt{\pm ab}}$, and then draw radii to the points of intersection of the curves, the angles made by these radii with the x axis of coordinates are the values of σ. Since the center of the circle is on the hyperbola, there are at least two intersections, and thus the equation in σ has at least two real roots. The geometrical construction readily affords the condition which a and b must satisfy in order that there may be four real roots. The condition is, that the length of the straight line drawn from the point a, b, on the hyperbola whose equation is $xy = \mathrm{ab}$, normal to the opposite branch, shall be less than unity. The equation to the normal which passes through the point x'', y'' on this curve is

$$x'' (x - x'') - y'' (y - y'') = 0 .$$

The condition that it passes through the point a, b gives,

$$x''(x''-\mathrm{a}) - y''(y''-\mathrm{b}) = 0\,, \quad x''y'' = \mathrm{ab}\,.$$

If we multiply the first of these by x''^2, we get

$$x''^3(x''-\mathrm{a}) - \mathrm{ab}(\mathrm{ab}-\mathrm{b}x'') = 0\,,$$

or, rejecting the useless factor $x''-\mathrm{a}$,

$$x''^3 + \mathrm{ab}^2 = 0\,,$$

whence $x'' = -\sqrt[3]{\mathrm{ab}^2}$, and by interchanging a and b, $y'' = -\sqrt[3]{\mathrm{a}^2\mathrm{b}}$. And thus the length of the normal

$$\begin{aligned}\sqrt{(x''-\mathrm{a})^2+(y''-\mathrm{b})^2} &= [(\mathrm{a}+\sqrt[3]{\mathrm{ab}^2})^2 + (\mathrm{b}+\sqrt[3]{\mathrm{a}^2\mathrm{b}})^2]^{\frac{1}{2}} \\ &= [\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}}]^{\frac{3}{2}}\,.\end{aligned}$$

Consequently if

$\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} < 1$, there will be four real roots;
$\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} = 1$, there will be four, and two will be equal;
$\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} > 1$, there will be only two real roots.

We will now show how to arrive at a direct solution of the problem by the employment of trigonometric formulas. If $\tan\sigma$ is taken for the unknown quantity, the equation, on which the solution of the problem depends, is

$$[c\cos\beta\tan\sigma + c\sin\beta]^2(1+\tan^2\sigma) = \tan^2\sigma\,,$$

or, if we put $\tan\sigma = x$,

$$(x+\tan\beta)^2(x^2+1) = \frac{x^2}{c^2\cos^2\beta}\,,$$

or, expanded,

$$x^4 + 2\tan\beta\,.\,x^3 + \frac{c^2-1}{c^2\cos^2\beta}x^2 + 2\tan\beta\,.\,x + \tan^2\beta = 0\,.$$

A quantity μ may be assumed such that this biquadratic shall be resolved into the two quadratics

$$x^2 + 2\frac{\sin\mu\cos(\beta+\mu)}{\cos\beta\cos 2\mu}x + \tan\beta\tan\mu = 0\,,$$
$$x^2 + 2\frac{\cos\mu\sin(\beta-\mu)}{\cos\beta\cos 2\mu}x + \tan\beta\cot\mu = 0\,.$$

That this is possible will be evident on multiplying the left hand members of these equations together, for, after some reductions easy to make, all the coefficients, with the exception of that of x^2, will be found to be identical

with those of the biquadratic; and, consequently, μ is determined by the equation

$$\tan\beta\,[\tan\mu + \cot\mu] + 2\,\frac{\sin 2\mu\,\sin(\beta-\mu)\,\cos(\beta+\mu)}{\cos^2\beta\,\cos^2 2\mu} = \frac{c^2-1}{c^2\cos^2\beta},$$

or,

$$\frac{c^2\sin 2\beta}{\sin 2\mu} - \frac{c^2\sin 2\mu\,[\sin 2\mu - \sin 2\beta]}{1-\sin^2 2\mu} = c^2 - 1,$$

or,

$$\sin^3 2\mu + (c^2-1)\sin 2\mu - c^2\sin 2\beta = 0.$$

That this cubic will always give at least one real value for μ, is evident on making in the left hand member $\sin 2\mu$ successively equal to -1, 0, 1; the results obtained are

$-c^2(1+\sin 2\beta)$, always negative;
$-c^2\sin 2\beta$, negative or positive, according to the sign of $\sin 2\beta$;
$+c^2(1-\sin 2\beta)$, always positive.

Moreover, it is plain that there is one real value of μ, which makes $\sin 2\mu$ and $\sin 2\beta$ have like signs; this value we shall adopt.

Making, according as c^2 is greater or less than unity, $c^2 = \sec^2\gamma$ or $c^2 = \cos^2\gamma'$, the above cubic is solved by these formulas (see Chauvenet's Trigonometry, p. 96), it being necessary to make three different cases.

Case I.

$$\tan\varphi = \frac{2\sin^2\gamma\tan\gamma}{\sqrt{27}\sin 2\beta}, \quad \tan\psi = \tan^{\frac{1}{3}}\frac{\varphi}{2}, \quad \sin 2\mu = \frac{2}{\sqrt{3}}\tan\gamma\cot 2\psi.$$

Case II.

$$\sin\varphi = \frac{2\sin\gamma'\tan^2\gamma'}{\sqrt{27}\sin 2\beta}, \quad \tan\psi = \tan^{\frac{1}{3}}\frac{\varphi}{2}, \quad \sin 2\mu = \frac{2}{\sqrt{3}}\sin\gamma'\,\mathrm{cosec}\,2\psi.$$

Case III.

$$\sin 3\varphi = \frac{\sqrt{27}\;\sin 2\beta}{2\sin\gamma'\tan^2\gamma'}, \quad \sin 2\mu = \frac{2}{\sqrt{3}}\sin\gamma'\sin(\varphi \pm 60°).$$

When ϕ is impossible in Case II, the formulas of Case III must be used; and the upper or lower member of the double sign in the second equation must be taken according as $\sin 2\beta$ is positive or negative, in order that $\sin 2\mu$ may have the same sign with $\sin 2\beta$. All the auxiliary angles ϕ, ψ and μ may be taken between the limits $\pm 90°$. Since $\sin 2\beta \sin 2\mu$ is always positive, $\tan\beta\tan\mu$ and $\tan\beta\cot\mu$ are so likewise, since they are respectively equivalent to

$$\frac{\sin 2\beta\sin 2\mu}{4\cos^2\beta\cos^2\mu} \quad\text{and}\quad \frac{\sin 2\beta\sin 2\mu}{4\cos^2\beta\sin^2\mu}.$$

Let us take two auxiliary angles θ and θ', determined by the equations

$$\sin 2\theta = -\frac{\tan^{\frac{1}{2}}\beta \tan^{\frac{1}{2}}\mu \cos\beta \cos 2\mu}{\sin\mu \cos(\beta+\mu)},$$

$$\sin 2\theta' = -\frac{\tan^{\frac{1}{2}}\beta \cot^{\frac{1}{2}}\mu \cos\beta \cos 2\mu}{\cos\mu \sin(\beta-\mu)},$$

or by the equations

$$\sin 2\theta = \mp \frac{\cos 2\mu}{\cos(\beta+\mu)} \sqrt{\frac{\sin 2\beta}{\sin 2\mu}},$$

$$\sin 2\theta' = \mp \frac{\cos 2\mu}{\sin(\beta-\mu)} \sqrt{\frac{\sin 2\beta}{\sin 2\mu}},$$

where the upper or the lower of the signs must be taken according as $\frac{\cos\beta}{\sin\mu}$ in the first and $\frac{\cos\beta}{\cos\mu}$ in the second are positive or negative; and 2θ and $2\theta'$ may also be taken within the limits $\pm$ 90°. The four values of x or $\tan\sigma$ are then

$$\tan\sigma = \tan^{\frac{1}{2}}\beta \tan^{\frac{1}{2}}\mu \tan\theta,$$
$$\tan\sigma = \tan^{\frac{1}{2}}\beta \tan^{\frac{1}{2}}\mu \cot\theta,$$
$$\tan\sigma = \tan^{\frac{1}{2}}\beta \cot^{\frac{1}{2}}\mu \tan\theta',$$
$$\tan\sigma = \tan^{\frac{1}{2}}\beta \cot^{\frac{1}{2}}\mu \cot\theta'.$$

If the value of $\sin 2\theta$ or of $\sin 2\theta'$ does not lie within the limits $\pm$ 1, it indicates that the two corresponding values of $\tan\sigma$ are imaginary. The ambiguity in the determination of σ from its tangent is to be removed by taking it in that quadrant which permits the equation

$$\sin 2\sigma = 2c \sin(\sigma + \beta)$$

to be satisfied.

Although all these roots will satisfy the equations with which we began this discussion, yet they do not all necessarily belong to the problem. The reason of this is, that the three equations are not a complete statement of all the conditions of the problem. If we denote by Δ the distance of the body, whose orbit we are determining, from the earth, we shall have

$$\Delta_{-1} = r\cos(\chi_0 - \eta - \lambda_{-1}) + R_{-1}\cos(\lambda_{-1} - L_{-1}),$$
$$\Delta_0 = r\cos(\chi_0 - \lambda_0) \quad + R_0 \cos(\lambda_0 - L_0),$$
$$\Delta_1 = r\cos(\chi_0 + \eta - \lambda_1) \quad + R_1\cos(\lambda_1 - L_1).$$

The conditions of the problem require that Δ_{-1}, Δ_0, Δ_1 shall be essentially positive. Hence, if any system of values of r, χ_0 and η renders any of these quantities negative, it must be rejected. These rejected solutions really belong to the problem when one or more of the quantities λ_{-1}, λ_0, λ_1 are increased by 180°. In fact, on referring to the equations with which we started, we see that they are not altered when any one of the quantities λ

is increased by 180°. The geometrical statement of the problem is more comprehensive than the application of it to the discovery of the elements of circular orbits. Instead of the above criteria for the rejection of solutions not applicable, the following, which is simpler, may be used, viz., that χ always must lie in the angle between $L + 180°$ and λ which is less than 180°.

This example is added for the sake of illustration:

Suppose, in the case of Venus revolving about the sun, we have these data,

Wash. Mean Time.	λ.	L.	$\log R$.
1869 Jan. 1.0	250° 22′ 59″.1	281° 24′ 54″.9	9.9926528
" June 15.0	94 37 54.9	84 33 34.1	0.0069342
" Nov. 27.0	292 3 21.2	245 32 49.3	9.9939666

There will be found

$$\log a_{-1} = 9.7048977_n, \quad \log a_0 = 9.2497072, \quad \log a_1 = 9.8545925,$$
$$\log k = 0.5426896, \quad \zeta = 324°\,41'\,4''.52, \quad \delta = 176°\,35'\,15''.25,$$
$$\log \mathrm{a} = 9.7678074_n, \quad \log \mathrm{b} = 9.3111404.$$

Constructing the equilateral hyperbola whose equation is $xy = -1$, and the circle whose radius is 2.89, and the coordinates of its center $x' = +1.69$, $y' = -0.59$, we find the two roots of the equation in σ, $\sigma = 7\frac{1}{2}°$, $\sigma = 241\frac{1}{2}°$. In fact, the value of $\mathrm{a}^2 + \mathrm{b}^2 = 1.0475$ shows that the equation has, in this case, but two real roots. Pursuing the calculation

$$\log c = 9.7928205, \quad \beta = 160°\,44'\,24''.60, \quad \gamma' = 51°\,38'\,20''.85.$$

Case II is to be used here.

$$\varphi = -50°\,40'\,40''.00, \quad \psi = -37°\,56'\,3''.23, \quad \mu = -34°\,30'\,27''.50, \quad \theta = 14°\,49'\,46''.36,$$

θ' is impossible, which confirms the preceding statement about the number of real roots; and the values of σ are

$$\sigma = 7°\,23'\,36''.95, \quad \sigma = 241°\,37'\,18''.04.$$

If we employ the tentative process with the equation

$$\sin 2\sigma = 2c \sin(\sigma + \beta),$$

we shall get $\sigma = 7°\,23'\,36''.97$ and $\sigma = 241°\,37'\,17''.95$; as these values are more accurate, we shall use them. The two solutions are

$$\omega = 1°\,16'\,6''.99, \quad \log r = 0.6767422, \quad \chi_0 = 272°\,29'\,17''.14, \quad \eta = 28°\,13'\,48''.02;$$
$$\omega = 197°\,31'\,54''.15, \quad \log r = 9.8624217, \quad \chi_0 = 108°\,45'\,4''.30, \quad \eta = 262°\,27'\,29''.00.$$

On applying the above-mentioned criteria, the first solution is seen to be inadmissible, it makes Δ_0 and Δ_1 negative. If both λ_0 and λ_1 are increased by 180°, the solution will apply. The given example has then but one solution.

Below we give a comparison between the values of the elements of Venus's orbit, as found in this example, and those of the "Tables"; the differences are of course to be attributed to the neglect of the eccentricity and inclination of the orbit, and in a smaller degree to aberration and perturbations.

	From the Example.	From the Table.
Mean Distance from the sun	0.7284868	0.7233323
Mean Longitude Jan. 1.0, 1869	206° 17′ 35″.30	204° 57′ 20″.89
Mean Motion in a Julian Year	2091552″.2	2106641″.438

MEMOIR No. 9.

New Method for Facilitating the Conversion of Longitudes and Latitudes of Heavenly Bodies, near the Ecliptic, into Right Ascensions and Declinations, and Vice Versa.

(Proceedings of the American Academy of Arts and Sciences, Vol. VIII, pp. 210–213, 1870.)

In the computation of a Lunar Ephemeris, the conversion of the longitudes and latitudes into right ascensions and declinations, forms no inconsiderable part of the work to be done. Prof. Hansen, at the end of his "*Tables de la Lune*," has given some tables, with the view of diminishing the amount of labor required in this conversion. But their employment seems to me to possess little, if any, advantage over the use of the ordinary formulas of spherical trigonometry. I propose the following method, which, perhaps in a slight degree, is more ready than that of Prof. Hansen.

Designating the right ascension, declination, longitude, latitude and the obliquity of the ecliptic respectively by α, δ, l, b and ε, we have the following equations:

$$\begin{aligned}\sin\delta &= \cos\varepsilon\sin b + \sin\varepsilon\cos b\sin l\\ &= \cos\varepsilon\sin b + \frac{\sin\varepsilon}{2}\sin(l+b) + \frac{\sin\varepsilon}{2}\sin(l-b),\end{aligned}$$

$$\tan\frac{\alpha+6^h}{2} = \frac{\cos\frac{\varepsilon+b+\delta}{2}}{\cos\frac{\varepsilon-(b+\delta)}{2}}\tan\frac{l+90^\circ}{2}.$$

The first equation is well known, the second is easily derived from the known formula, expressed in the usual notation,

$$\tan\frac{A}{2}\tan\frac{B}{2} = \frac{\sin(s-c)}{\sin s},$$

when we remember that, in considering the triangle formed by the heavenly body and the poles of the equator and ecliptic, A, B, s and c are replaced by $90^\circ+\alpha$, $90^\circ-l$, $90^\circ+\frac{\varepsilon-(b+\delta)}{2}$ and ε.

Suppose we were to tabulate the functions $\cos\varepsilon\sin A$ and $\frac{\sin\varepsilon}{2}\sin A$ for a certain value of ε (as 23° 27′ 20″ which is nearly its value at present), and in small side tables put the variations of these functions for increments

of $1''$, $2''$, ..., $9''$ in ε; we should have the value of $\sin\delta$ by entering the first table with the argument $A=b$, and the second successively with the arguments $A=l+b$ and $A=l-b$, and adding the results thus obtained, after having corrected them for the deviation of the value of ε from that adopted in the tables. After which the value of δ could be obtained from a table of natural sines. For the case of the moon, the first function would need tabulation only between the limits $A=0°$ and $A=5°\,17'$; it might be tabulated for every $10''$. The second would have to be tabulated from $0°$ to $90°$; it might be given for every minute of arc. The numbers in these tables might be rendered always positive by adding a constant to them; as, for instance, 0.1 to the first function, and 0.2 to the second; and thus the addition of the three terms of $\sin\delta$ be made easier. We should then have to subtract 0.5 from the sum, in order to get $\sin\delta$; or we might prepare a special table, which, with the argument $0.5+\sin\delta$, should give δ. But, by the addition of these constants, the extent of the tables would be doubled, as it would be necessary to tabulate the numbers which correspond to negative values of the arguments.

The factor by which $\tan\dfrac{l+90°}{2}$ must be multiplied to obtain $\tan\dfrac{\alpha+6^h}{2}$ is always positive, and, ε being regarded as constant, is a function of $b+\delta$, and, for negative values of $b+\delta$, its value is the reciprocal of that which corresponds to positive values of $b+\delta$. Moreover, when $b+\delta$ is a tolerably small angle, it does not differ much from unity, and varies very uniformly. In the case of the moon, $b+\delta$ rarely exceeds the limits $\pm 34°$, and the common logarithm of this quantity lies between 9.9447979 and 0.0552021; and its rate of change per minute of arc in $b+\delta$ varies only from 262 to 289 units of the seventh decimal place. We may, with the better advantage, tabulate the function

$$\log\cos\frac{\varepsilon-A}{2}-\log\cos\frac{\varepsilon+A}{2},$$

for every minute of arc of the argument A from $0°$ to $34°$, with the precept that it is to be subtracted from $\log\tan\dfrac{l+90°}{2}$ when $b+\delta$ is a positive angle, but added when $b+\delta$ is negative. It will be necessary to append to the table the variation of the function for a change in ε. The functions $\log\tan\left(45°+\dfrac{l}{2}\right)$ and $\log\tan\left(45°+\dfrac{\alpha}{2}\right)$ can be found from the logarithmic tables, but some labor would be spared had we tables which gave $\log\tan\left(45°+\dfrac{A}{2}\right)$ with the argument A both in arc and time; which

tables would be useful in many other cases, since this function is frequently met with in trigonometric formulas.

The modifications necessary in applying this method to the inverse problem of determining the longitude and latitude from the right ascension and declination are obvious. The variations due to the change of the obliquity might perhaps be neglected in using the tables, especially in the case of the declination, and computed at the end by means of the very simple formulas

$$\frac{d\alpha}{d\varepsilon} = -\tan\delta\cos\alpha\,, \quad \frac{d\delta}{d\varepsilon} = \sin\alpha\,.$$

Take this example for illustration:

On Jan. 14.0, 1871, G. M. T., we have in the case of the moon,

$l = 206°\,40'\,35''.9$,	$\varepsilon = 23°\,27'\,19''.81$	
$b = +5°\;\;3'\,16''.0$	From Tab. I., Arg. b,	$+0.0808224$
	$-\;\;1.7 \times (\varDelta\varepsilon = -0.19)$	0
$l + b = 211°\,43'\,51''.9$	From Tab. II., Arg. $l + b$,	-0.1046706
	$-11.7 \times \varDelta\varepsilon$	$+2$
$l - b = 201°\,37'\,19''.9$	From Tab. II., Arg. $l - b$,	-0.0733354
	$-\;\;8.2 \times \varDelta\varepsilon$	$+2$
$\delta = -5°\,34'\,37''.16$	$\sin\delta$	-0.0971832
$b + \delta = -0°\,31'\,21''.16$	$\log\tan 148°\,20'\,17''.95$	9.7900662_n
$\alpha = \;13^h\,46^m\,19^s.12$	From Tab. III., to be added	0.0008223
	$+\,0.09 \times \varDelta\varepsilon$	0
	$\log\tan 9^h\,53^m\,9^s.56$	9.7908885_n

The objection to this method is, that so many arguments $l + b$, $l - b$, $b + \delta$, $45° + \frac{l}{2}$, and α from $45° + \frac{\alpha}{2}$ are to be formed; but this is confessedly less fatiguing than the taking of tabular quantities from a table.

It may be allowed to notice here a series, which determines α in terms of l, viz.,

$$\alpha = l + \frac{2}{1}\tan\frac{\varepsilon}{2}\tan\frac{b+\delta}{2}\cos l - \frac{2}{2}\tan^2\frac{\varepsilon}{2}\tan^2\frac{b+\delta}{2}\sin 2l$$
$$- \frac{2}{3}\tan^3\frac{\varepsilon}{2}\tan^3\frac{b+\delta}{2}\cos 3l + \frac{2}{4}\tan^4\frac{\varepsilon}{2}\tan^4\frac{b+\delta}{2}\sin 4l - \text{etc.}$$

As $\tan\frac{\varepsilon}{2}\tan\frac{b+\delta}{2}$, in the case of the Moon, is always between the limits $\pm\frac{2}{31}$, the above series is, for this body, quite convergent.*

* This series under a slightly different form is given in the memoir of Lagrange entitled *Solutions de quelques Problèmes d'Astronomie Sphérique par moyen des Séries* (See *Œuvres*, Tom. IV, p. 293).

I add the values of the function $\log \dfrac{\cos \dfrac{\varepsilon - A}{2}}{\cos \dfrac{\varepsilon + A}{2}}$, computed for every degree from 0° to 35° of the argument A and for $\varepsilon = 23° \, 27' \, 20''$.

A	$\log \dfrac{\cos \frac{\varepsilon - A}{2}}{\cos \frac{\varepsilon + A}{2}}$	Δ	Δ^2	Change of this function for an incr. of 1″ in ε in units of the seventh decimal.
0°	.0000000	+15736		+0.00
1	.0015736	15738	+ 2	0.19
2	.0031474	15744	6	0.38
3	.0047218	15751	7	0.57
4	.0062969	15761	10	0.77
5	.0078730	15773	12	0.96
6	.0094503	15789	16	1.15
7	.0110292	15806	17	1.34
8	.0126098	15826	20	1.54
9	.0141924	15849	23	1.73
10	.0157773	15874	25	1.92
11	.0173647	15902	28	2.12
12	.0189549	15933	31	2.31
13	.0205482	15965	32	2.50
14	.0221447	16002	37	2.70
15	.0237449	16040	38	2.89
16	.0253489	16081	41	3.09
17	.0269570	16124	43	3.28
18	.0285694	16172	48	3.48
19	.0301866	16221	49	3.68
20	.0318087	16273	52	3.88
21	.0334360	16328	55	4.08
22	.0350688	16386	58	4.28
23	.0367074	16447	61	4.48
24	.0383521	16511	64	4.68
25	.0400032	16578	67	4.88
26	.0416610	16648	70	5.08
27	.0433258	16721	73	5.29
28	.0449979	16797	76	5.49
29	.0466776	16877	80	5.70
30	.0483653	16959	82	5.90
31	.0500612	17046	87	6.11
32	.0517658	17135	89	6.32
33	.0534793	17228	93	6.53
34	.0552021	+17325	+97	6.74
35	.0569346			+6.95

MEMOIR No. 10.

Correction of the Elements of the Orbit of Venus.

(Extracted from Tables of Venus, prepared for the use of the American Ephemeris and Nautical Almanac, Washington, 1872.)

The elements, adopted for comparison with observation, are, in the main, those on which Leverrier has based his tables. They are

Epoch, 1850, Jan. 1.0, Paris Mean Time.

$$
\begin{aligned}
L' &= 245^\circ\ 33'\ 14''.70\\
\pi' &= 129\ \ 27\ \ 14.5\\
\Omega' &= 75\ \ 19\ \ 52.3\\
i' &= 3\ \ 23\ \ 34.83\\
e' &= 0.00684331\\
n' &= 2106641''.3831
\end{aligned}
$$

The value of n' has been changed in order to make the adopted tropical motion coincide with Leverrier's value. The values of the disturbing masses are

Mercury $m = \frac{1}{4865751}$, Venus $m' = \frac{1}{408134}$, Earth $m'' = \frac{1}{322800}$,
Mars $m''' = \frac{1}{3200900}$, Jupiter $m^{\mathrm{IV}} = \frac{1}{1050}$, Saturn $m^{\mathrm{V}} = \frac{1}{3500}$.

The mass of Mercury is that of Encke,* the mass of the Earth and Moon is that found by Prof. S. Newcomb,† and which corresponds to the value $8''.848$ of the mean horizontal parallax of the Sun; the values of the other masses are those adopted by Hansen and Olufsen. On these values of the disturbing masses depend the expressions of the secular and periodic perturbations used. The true longitude of the Sun is derived from the apparent longitude of Hansen's and Olufsen's *Tables du Soleil* by subtracting the effect of aberration corresponding to the constant $20''.255$.

All the elements, except the mean motion, are determined, with nearly all the precision possible by the modern observations; that is to say, those comprehended in the interval from 1836 up to the present time. The addition of the observations made previously to 1836 to the discussion, would scarcely

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* Astronomische Nachrichten, No. 443.

† Astronomical and Meteorological Observations made at the United States Naval Observatory during the year 1865. Appendix II, p. 29.

increase this precision. For the mean motion, we must employ ancient observations; and for this purpose it seems better to depend on the data furnished by the Transits of 1761 and 1769 than on the somewhat uncertain observations of Bradley.

Encke's reduction of these Transits, corrected to conform with the positions of the Sun derived from the *Tables du Soleil*, will be adopted. All the longitudes mentioned here are referred to the mean equinox of date.

For the Transit of 1761 Encke gives

Paris Mean Time	$=$ 1761, June 5^d 17^h 30^m
True Longitude of the Sun	$= 75° 35' 49''.6$
Latitude of the Sun	$= +0.6$
Orbit Longitude of Venus	$= 255\ 35\ 34.45$
Heliocentric Latitude of Venus	$= -3\ 45.91$

But the *Tables du Soleil* give $75° 35' 52''.05$ and $+0''.53$ as the longitude and latitude of the Sun. Consequently, the adopted position of Venus is

Orbit Longitude $= 255° 35' 36''.90$, Heliocentric Latitude $= -3' 45''.84$.

For the Transit of 1769 Encke gives

Paris Mean Time	$=$ 1769, June 3^d 10^h 10^m
True Longitude of the Sun	$= 73° 27' 13''.8$
Latitude of the Sun	$= 0.0$
Orbit Longitude of Venus	$= 253\ 27\ 13.17$
Heliocentric Latitude of Venus	$= +4\ 4.56$

The *Tables du Soleil* give $73° 27' 14''.25$ and $+0''.04$ as the longitude and latitude of the Sun. Consequently, the adopted position of Venus is

Orbit Longitude $= 253° 27' 13''.62$, Heliocentric Latitude $= +4' 4''.52$.

The meridian observations have been corrected to conform with the constant $8''.848$ of solar parallax, and to the following expression for the semi-diameter:

$$\frac{8''.546}{\Delta} + 0''.57\,.$$

In other respects Leverrier's reduction has been adopted. With regard to the Greenwich and Paris observations which have accumulated since Leverrier made his investigation, that is, from 1858 forward, as a comparison of the places, given in the annual volumes, for the fundamental stars, with Dr. Gould's *Standard Places*, showed no sensible *average* difference in the right ascensions, no correction for difference of equinoxes has been applied.

To the Washington observations in declination in the years 1866, 1867, has been applied the correction + 0″.75. (See *Washington Observations for* 1867, Appendix III, pp. 20, 21.)

In forming the following normals, Paris observations have been combined with Greenwich; but Washington observations have been kept separate. The normals, formed from the latter, are those given for Washington Mean Noon. The Paris observations used are not in great number, and belong to the years 1838 and 1856–1866. The comparisons are Obs.—Cal.

Normals in the inferior part of the Orbit.

No.	Greenwich M. T.			App. R. A.	App. Dec.	No. Obs.	Δ α	Δ δ
			d	h m s	° ′ ″		s	″
1	1836,	June	9.0	8 16 6.380	+21 53 40.12	4	+0.082	+0.62
2		July	2.0	8 52 43.140	+16 16 11.35	5	—0.057	+0.63
3		July	13.0	8 43 59.799	+14 17 35.12	4	—0.054	—0.32
4		Aug.	7.0	7 47 48.091	+13 41 44.35	3	+0.228	—0.60
5		Aug.	30.0	7 56 5.580	+15 11 1.98	4	+0.083	—1.71
6	1838,	Jan.	12.0	22 36 4.483	— 8 23 42.65	7	+0.079	+0.48
7		Feb.	2.0	23 19 4.936	— 0 5 1.83	5	—0.163	+5.07
8		Feb.	22.0	23 11 48.498	+ 3 26 16.93	3	+0.050	+2.00
9		Mar.	12.0	22 33 39.400	— 0 1 38.66	3	+0.178	+1.75
10		Mar.	24.0	22 23 12.226	— 3 12 55.39	10	+0.111	—1.18
11		April	7.0	22 37 31.008	— 4 49 5.56	13	+0.096	—1.02
12	1839,	Sept.	21.0	12 58 21.552	—14 51 58.87	4	—0.147	—0.66
13		Oct.	12.0	12 19 41.626	— 9 44 43.41	9	+0.047	+0.82
14	1841,	May	1.0	3 50 40.864	+25 34 44.55	6	+0.009	+0.39
15		May	27.0	2 59 23.728	+17 13 40.10	5	+0.254	+1.95
16		June	12.0	2 59 45.783	+14 23 34.07	4	—0.021	—0.92
17	1842,	Dec.	15.0	17 56 8.706	—22 32 23.92	5	—0.140	+2.34
18	1843,	Jan.	10.0	17 15 35.705	—17 35 57.26	2	—0.042	+0.29
19	1844,	May	31.0	7 46 25.585	+23 55 35.09	6	—0.047	+1.23
20		July	30.0	7 49 49.182	+13 59 37.34	6	—0.046	—0.68
21	1846,	Jan.	16.0	22 44 36.217	— 6 45 4.41	3	—0.074	+0.13
22		Feb.	8.0	23 14 37.585	+ 1 8 50.95	4	—0.092	+0.20
23		Mar.	18.0	22 15 8.390	— 3 5 52.55	2	+0.210	—3.17
24	1847,	Aug.	15.0	12 16 12.840	— 4 47 4.42	4	+0.052	+0.69
25		Sept.	23.0	12 43 32.402	—13 41 51.42	4	—0.203	+0.64
26		Nov.	15.0	12 36 33.246	— 3 37 36.16	5	+0.206	—0.82
27	1849,	May	2.0	3 36 3.678	+24 41 22.91	5	+0.187	+0.09
28		June	8.0	2 49 10.035	+14 4 37.76	10	—0.087	+3.60
29	1850,	Nov.	23.0	18 8 47.037	—26 55 13.20	3	—0.159	—2.65
30		Dec.	17.0	17 33 52.085	—21 38 46.95	2	—0.033	—0.47
31	1851,	Jan.	20.0	17 20 48.470	—17 41 28.31	4	+0.296	—0.76
32	1852,	July	10.0	8 24 23.066	+15 40 33.60	9	+0.040	+0.44
33		Aug.	16.0	7 25 42.850	+15 24 38.50	4	+0.182	—0.78
34		Sept.	6.0	8 4 29.218	+15 58 35.33	4	+0.126	—1.14
35	1854,	Jan.	20.0	22 50 6.361	— 5 16 4.06	6	+0.061	—0.04
36		Feb.	3.0	23 6 4.528	— 0 34 39.68	3	+0.042	+1.67
37		Feb.	20.0	22 49 33.074	+ 1 19 46.87	5	+0.221	+0.45
38	1855,	Aug.	18.0	12 20 36.824	— 5 57 52.34	7	+0.007	+0.32
39		Sept.	20.0	12 35 48.073	—12 52 25.49	5	+0.120	+2.32

No.	Greenwich M. T.		App. R. A.	App. Dec.	No. Obs.	Δ α	Δ δ
		d	h m s	° ′ ″		s	″
40	1855, Oct.	12.0	11 55 6.943	− 6 28 37.38	4	+0.076	−1.62
41	Nov.	16.0	12 36 57.050	− 3 23 38.44	5	+0.148	−0.90
42	1857, Feb.	16.0	0 49 21.025	+ 6 27 10.65	13	+0.063	+0.46
43	Mar.	18.0	2 36 3.575	+19 31 12.35	5	−0.058	−0.37
44	April	16.0	3 35 55.521	+25 33 57.52	7	+0.027	−0.85
45	May	21.0	2 42 50.763	+16 59 35.65	8	+0.118	+0.56
46	June	13.0	2 50 26.725	+13 31 18.27	13	+0.020	+0.81
47	June	26.0	3 20 49.536	+14 46 46.18	12	+0.084	+1.07
48	1858, Aug.	17.0	12 21 4.321	− 2 10 32.51	9	−0.139	+1.03
49	Sept.	18.0	14 31 57.511	−17 24 17.46	4	−0.058	−1.56
50	Oct.	10.0	16 2 1.666	−24 42 17.26	10	−0.086	−0.62
51	Nov.	7.0	17 37 19.047	−28 1 51.96	11	+0.050	−3.24
52	Nov.	29.0	17 55 9.651	−25 54 31.11	3	+0.311	−4.70
53	Dec.	21.0	17 7 52.455	−20 4 43.46	4	+0.203	−2.23
54	1859, Jan.	10.0	16 58 27.618	−17 24 53.14	7	+0.051	+3.60
55	Jan.	29.0	17 40 25.353	−18 26 8.24	8	+0.138	+0.17
56	1860, May	3.0	5 53 18.564	+26 36 37.27	4	+0.034	+1.43
57	May	23.0	7 16 2.843	+25 23 36.95	5	+0.042	+1.53
58	June	19.0	8 23 55.823	+19 58 30.44	5	+0.103	+2.43
59	July	10.0	8 11 15.899	+16 8 22.57	6	+0.103	+2.50
60	Aug.	31.0	7 48 10.699	+16 21 14.53	7	+0.203	+0.18
61	Sept.	22.0	9 1 57.720	+14 41 24.01	11	+0.174	−0.67
62	1861, Dec.	10.0	20 34 32.810	−21 9 42.34	4	−0.020	−1.44
63	Dec.	26.0	21 37 51.853	−15 29 11.52	7	+0.036	−1.41
64	1862, Jan.	16.0	22 38 24.381	− 6 59 2.66	9	+0.063	−0.43
65	Feb.	12.0	22 50 59.987	+ 0 17 57.58	2	+0.201	−2.41
66	Mar.	11.0	21 58 59.897	− 3 59 31.67	5	+0.211	+3.72
67	April	23.0	23 14 6.685	− 4 20 27.06	9	+0.061	+0.09
68	May	13.0	0 26 3.479	+ 1 19 59.00	4	−0.069	+2.83
69	1863, July	11.0	10 24 37.937	+10 53 34.89	7	−0.014	+0.74
70	Aug.	1.0	11 35 5.496	+ 1 22 1.05	6	−0.004	−2.34
71	Aug.	12.0	12 4 25.882	− 3 26 49.57	7	+0.106	−4.25
72	Sept.	1.0	12 35 55.785	−10 23 46.78	6	−0.108	+0.38
73	Sept.	19.0	12 24 54.206	−11 49 46.67	6	+0.117	+1.85
74	Oct.	28.0	11 50 36.106	− 1 43 20.76	2	+0.117	−3.63
75	Nov.	20.0	12 47 33.271	− 3 53 37.08	5	+0.202	−2.15
76	1865, Feb.	13.0	0 38 41.720	+ 5 8 4.61	4	−0.042	−1.17
77	Mar.	25.0	2 54 9.362	+21 44 33.17	7	−0.008	+0.69
78	April	9.0	3 23 1.559	+24 46 29.99	10	+0.057	+0.96
79	April	25.0	3 20 45.353	+24 31 33.59	11	+0.102	−0.08
80	May	7.0	2 56 28.204	+21 8 3.49	7	+0.201	+1.48
81	May	24.0	2 28 59.092	+14 58 17.96	9	+0.233	+1.59
82	June	11.0	2 42 18.347	+13 4 19.64	8	+0.208	+0.26
83	June	22.0	3 7 28.235	+14 5 20.47	7	+0.139	−0.07
84	July	11.0	4 9 16.618	+17 22 1.01	9	+0.106	+0.31
85	1866, Sept.	25.0	15 1 14.407	−20 15 30.52	3	+0.066	+0.55
86	Oct.	16.0	16 24 17.041	−26 5 30.30	7	+0.047	−1.53
87	Oct.	27.0	17 2 22.875	−27 36 58.50	3	−0.002	+0.46
88	Nov.	15.0	17 44 28.584	−27 42 20.26	9	+0.208	−0.31
89	Nov.	30.0	17 39 2.404	−25 25 42.81	4	+0.417	+0.05
90	Dec.	28.0	16 44 36.668	−18 5 53.53	2	+0.359	+0.19
91	1867, Feb.	7.0	18 9 47.537	−19 2 58.09	6	+0.174	+1.04
92	Mar.	30.0	21 52 48.772	−12 51 21.47	2	+0.045	−0.29

No.	Greenwich M. T.	App. R. A.	App. Dec.	No. Obs.	Δ α	Δ δ
	d	h m s	° ′ ″		s	″
93	1868, May 6.0	6 7 11.834	+26 42 54.56	6	−0.111	+0.76
94	May 19.0	7 0 21.501	+25 57 6.61	3	+0.054	+0.95
95	May 29.0	7 34 52.419	+24 29 40.91	4	+0.103	+0.74
96	June 12.0	8 8 50.822	+21 41 50.55	9	+0.059	+0.79
97	June 29.0	8 16 30.999	+18 13 10.72	7	+0.203	−0.09
98	July 14.0	7 47 9.381	+16 9 52.23	6	+0.177	+1.23
99	July 28.0	7 14 21.065	+15 34 15.57	4	+0.173	−1.35
100	Aug. 15.0	7 11 41.426	+16 13 52.99	1	+0.050	−0.96
101	Aug. 26.0	7 32 23.633	+16 38 18.77	4	+0.104	+0.03
102	Sept. 4.0	7 57 43.509	+16 35 5.21	4	−0.069	−0.37
103	Sept. 18.0	8 46 43.419	+15 25 27.66	6	+0.004	−0.63
104	1869, Dec. 1.0	19 55 0.743	−23 33 19.35	2	+0.033	+0.09
105	Dec. 23.0	21 26 28.614	−16 37 52.39	1	+0.038	+0.94
106	1870, Jan. 3.0	22 2 46.594	−12 18 0.76	1	+0.092	+1.72
107	Jan. 27.0	22 48 13.855	− 3 24 26.46	4	+0.339	+2.64
108	Feb. 21.0	22 19 12.992	− 1 14 10.51	3	+0.257	+2.49
109	Mar. 19.0	21 49 6.160	− 6 42 37.21	2	+0.195	+1.21
110	April 5.0	22 18 34.856	− 7 18 36.33	2	+0.087	+2.55
111	April 12.0	22 37 59.072	− 6 36 2.61	3	+0.266	+2.67
112	April 22.0	23 10 0.139	− 4 46 21.21	3	+0.060	+2.34
113	May 23.0	1 5 55.518	+ 4 54 43.03	7	+0.036	+0.63
114	June 13.0	2 33 32.558	+12 34 56.06	4	+0.148	−0.12
115	July 13.0	4 53 4.820	+20 50 51.48	5	+0.047	+1.00
116	Aug. 8.0	7 5 26.170	+22 6 39.93	3	−0.032	+1.00
117	Aug. 25.0	8 32 16.460	+19 12 2.24	5	−0.107	+0.34
118	Sept. 15.0	10 14 53.930	+12 4 3.63	2	−0.060	+0.89
119	Sept. 26.0	11 6 24.896	+ 7 13 14.54	5	−0.026	+0.78
120	Oct. 12.0	12 19 50.532	− 0 31 51.43	5	−0.170	+1.03
121	Nov. 1.0	13 52 35.280	−10 14 46.16	4	−0.157	−0.24
122	Nov. 18.0	15 15 37.167	−17 20 12.09	3	−0.012	+1.06
123	Dec. 24.0	18 28 7.770	−23 56 17.07	1	−0.033	−1.33
124	1871, Jan. 4.0	19 28 19.110	−22 55 7.73	1	−0.029	+1.42
	Washington M. T.					
125	1863, Aug. 19.0	12 19 46.295	− 6 19 30.41	13	+0.078	+1.33
126	Sept. 12.0	12 34 6.510	−12 5 24.23	9	+0.071	+1.08
127	Oct. 19.0	11 42 32.127	− 2 57 57.34	10	+0.236	−0.94
128	Nov. 15.0	12 32 43.563	− 2 56 9.17	11	+0.071	+0.35
129	1865, Feb. 7.0	0 16 19.073	+ 2 10 43.90	6	+0.034	+0.11
130	Feb. 23.0	1 16 59.279	+10 9 15.84	4	−0.039	+0.44
131	Mar. 11.0	2 13 1.700	+17 6 32.28	8	+0.037	+1.02
132	Mar. 28.0	3 2 8.034	+22 35 31.27	3	−0.113	+1.20
133	April 18.0	3 26 40.672	+25 11 16.40	6	+0.010	+1.81
134	May 2.0	3 7 27.361	+22 47 40.20	4	+0.240	+0.61
135	May 18.0	2 34 22.761	+16 46 48.36	7	+0.091	+1.46
136	June 4.0	2 32 28.520	+13 9 9.64	8	+0.095	+0.42
137	June 26.0	3 19 28.164	+14 43 23.83	9	+0.079	0.00
138	July 20.0	4 45 13.373	+18 57 33.95	8	+0.080	+1.11
139	1866, Sept. 12.0	14 9 6.234	−15 7 32.81	5	−0.062	−0.62
140	Oct. 6.0	15 46 27.321	−23 49 35.82	4	−0.031	−0.75
141	Oct. 19.0	16 36 2.952	−26 39 2.71	8	+0.061	−2.11
142	Nov. 9.0	17 36 3.668	−28 1 18.33	7	+0.134	−1.56
143	Nov. 28.0	17 42 2.599	−25 52 0.26	6	+0.385	−1.88
144	Dec. 19.0	16 55 50.052	−20 0 53.68	2	+0.570	+2.20
145	1867, Jan. 22.0	17 17 25.170	−17 59 43.41	10	+0.222	−0.14

Normals in the superior part of the Orbit.

No.	Greenwich M. T.			App. R. A.	App. Dec.	No. Obs.	$\Delta\alpha$	$\Delta\delta$
				h m s	° ′ ″		s	″
146	1858,	Jan.	23.0	19 46 16.637	−21 53 48.46	3	+0.022	−2.26
147		April	23.0	2 56 59.252	+16 35 27.79	5	−0.005	−0.10
148		June	14.0	7 27 55.977	+23 33 18.26	13	+0.078	−0.13
149		July	19.0	10 17 52.788	+12 10 47.22	5	−0.035	+0.11
150	1859,	Feb.	23.0	19 14 56.589	−19 15 37.66	7	+0.022	−0.82
151		Mar.	18.0	20 57 4.220	−16 11 30.29	6	+0.188	+2.39
152		June	17.0	3 46 35.988	+18 29 1.65	4	+0.033	−0.40
153		July	19.0	6 31 41.515	+23 6 57.50	11	−0.021	−0.31
154		Aug.	23.0	9 32 0.652	+15 49 11.42	8	−0.016	−0.44
155		Nov.	13.0	16 1 34.918	−20 45 38.42	5	+0.044	−1.75
156		Dec.	17.0	19 5 56.987	−23 55 27.75	4	+0.043	−3.39
157	1860,	Jan.	17.0	21 46 14.280	−15 10 47.89	5	+0.016	−2.66
158		Feb.	29.0	1 0 24.170	+ 6 15 55.78	3	−0.062	−0.77
159		April	19.0	4 48 31.013	+25 10 19.92	4	−0.014	−0.04
160		Oct.	24.0	11 13 17.826	+ 5 51 46.41	5	+0.086	−0.82
161		Dec.	10.0	14 42 35.914	−13 40 52.46	5	−0.060	−0.81
162	1867,	May	14.0	1 11 9.973	+ 5 34 41.60	6	+0.113	+0.44
163		June	17.0	3 49 8.762	+18 38 58.87	5	+0.050	+1.11
164		Aug.	18.0	9 10 1.066	+17 23 44.04	6	−0.059	+0.60
165		Oct.	15.0	13 41 6.075	− 9 28 37.18	4	+0.009	−1.01
166		Nov.	19.0	16 36 3.118	−22 25 34.51	5	−0.007	−0.51
167	1868,	Oct.	16.0	10 40 43.471	+ 8 38 39.82	9	+0.100	+0.01
168		Dec.	17.0	15 18 46.956	−16 23 36.95	6	+0.083	+0.83
169	1869,	Jan.	12.0	17 32 57.506	−22 22 25.57	5	+0.050	−1.48
170		April	20.0	1 36 7.195	+ 8 43 59.95	6	−0.070	+0.55
171		June	17.0	6 29 55.784	+24 7 55.16	5	−0.020	+0.32
172		July	16.0	9 1 6.090	+18 33 2.87	4	−0.208	+0.78
173		Aug.	26.0	12 10 0.084	− 0 7 34.79	5	−0.010	+0.29
174		Sept.	21.0	14 5 26.833	−13 7 17.72	4	−0.183	+1.02
175		Oct.	13.0	15 49 46.368	−21 42 44.87	5	−0.026	+1.43

In order to have as few unknown quantities as possible in the equations of condition, the differences $\Delta\alpha$ and $\Delta\delta$ have been changed into $\cos\eta.\Delta\theta$ and $\Delta\eta$; θ denoting the geocentric longitude of Venus referred to a plane drawn through the center of the Earth parallel to the plane of the orbit of Venus, and η denoting the corresponding latitude. The formulas used are given in Watson's *Theoretical Astronomy*, pp. 153–159.

In the following equations we have put

$$x = \Delta L_0' - 2\sin^2\frac{i'}{2}\Delta\Omega', \quad y = 100\Delta n', \quad z = \Delta e', \quad u = e'\left(\Delta\pi' - 2\sin^2\frac{i'}{2}\Delta\Omega'\right),$$

all expressed in seconds of arc; and x', y', z' and u' denote the corresponding quantities in reference to the solar elements. In the computation of the coefficients of the last, roughly approximate formulas have been used.

A mean of the Transits of 1761 and 1769 gives

$$+0.992x - 0.839y + 1.61z + 1.17u + 1.00x' - 0.84y' + 0.83z' - 1.82u' = +1''.745.$$

The indeterminate correction of the Sun's semi-diameter nearly disappears from this mean.

The following equations of condition are numbered with the same numbers as the normals from which they are derived. The last column contains the residuals which remain after the elements have been corrected as shown in the sequel.

Equations of Condition.

No.										Residuals.
1	$-0.40x$	$+0.05y$	$-0.36z$	$-1.44u$	$+1.43x'$	$-0.19y'$	$-0.21z'$	$-3.06u'$	$= +1''.01$	$+0''.97$
2	−1.37	+0.18	−0.87	−2.97	+2.41	−0.32	−1.45	−4.69	= −0.95	−1.02
3	−2.05	+0.28	−0.87	−4.16	+3.08	−0.41	−2.17	−5.74	= −0.69	−0.74
4	−2.07	+0.28	−0.02	−4.28	+3.11	−0.41	−2.65	−5.57	= +3.37	+3.37
5	−0.80	+0.11	+0.31	−2.15	+1.80	−0.24	−2.22	−3.16	= +1.44	+1.43
6	−0.31	+0.04	−0.42	+1.41	+1.30	−0.16	+1.86	+2.32	= +1.27	+0.68
7	−0.98	+0.12	−0.93	+2.31	+1.98	−0.24	+3.56	+2.59	= −0.23	−1.25
8	−2.27	+0.27	−2.31	+4.06	+3.27	−0.39	+5.84	+3.12	= +1.48	−0.38
9	−2.44	+0.29	−3.04	+3.85	+3.40	−0.40	+6.26	+2.85	= +3.13	+1.22
10	−1.70	+0.20	−2.53	+2.69	+2.70	−0.32	+5.18	+2.00	= +1.13	−0.27
11	−0.90	+0.11	−1.76	+1.56	+1.91	−0.22	+3.95	+1.01	= +0.96	+0.08
12	−2.06	+0.21	+3.53	−2.38	+3.08	−0.32	−6.12	−0.14	= −1.66	−2.07
13	−2.51	+0.26	+4.64	−2.02	+3.51	−0.36	−7.01	+0.39	= +0.29	−0.34
14	−2.00	+0.17	−4.12	−0.54	+3.00	−0.26	+4.57	−3.99	= +0.22	−0.91
15	−2.09	+0.18	−4.05	−1.48	+3.10	−0.27	+4.28	−4.47	= +4.08	+3.09
16	−1.12	+0.10	−2.39	−1.18	+2.12	−0.14	+2.72	−3.49	= −0.59	−1.14
17	−2.69	+0.19	+3.74	+3.87	+3.69	−0.26	−1.82	+7.26	= −2.09	−4.20
18	−1.58	+0.11	+1.80	+2.98	+2.58	−0.18	−0.65	+5.38	= −0.63	−2.10
19	−0.27	+0.01	−0.63	−1.18	+1.27	−0.07	+0.21	−2.76	= −0.81	−0.95
20	−2.40	+0.13	−0.41	−4.82	+3.40	−0.18	−2.47	+6.18	= −0.57	−1.14
21	−0.47	+0.02	−0.47	+1.64	+1.47	−0.06	+2.19	+2.48	= −0.98	−1.79
22	−1.54	+0.06	−1.31	+3.16	+2.54	−0.10	+4.37	+3.00	= −1.19	−2.93
23	−1.95	+0.07	−2.61	+3.15	+2.95	−0.11	+5.79	+2.51	= +1.84	−0.20
24	−0.40	+0.01	+1.03	−1.15	+1.40	−0.03	−2.79	−1.18	= +0.42	+0.24
25	−2.29	+0.05	+3.87	−2.60	+3.28	−0.07	−6.58	−0.29	= −2.95	−3.92
26	−0.55	+0.01	+1.78	−0.38	+1.55	−0.03	−3.33	+1.04	= +3.15	+2.75
27	−2.22	+0.02	−4.51	−0.53	+3.22	−0.02	+4.94	−4.09	= +2.47	+0.72
28	−1.22	+0.01	−2.59	−1.14	+2.22	−0.01	+3.02	−3.46	= +0.04	−0.85
29	−1.55	−0.01	+2.84	+1.97	+2.55	+0.02	−2.07	+4.94	= −2.24	−3.81
30	−2.73	−0.03	+3.75	+3.93	+3.72	+0.04	−2.06	+7.28	= −0.41	−3.18
31	−0.88	−0.01	+1.17	+2.02	+1.88	+0.02	−1.96	+4.10	= +4.28	+3.11
32	−2.18	−0.05	−1.24	−4.31	+3.18	+0.08	−1.77	−6.06	= −0.38	−1.64
33	−1.26	−0.03	+0.04	−2.92	+2.26	+0.06	−2.00	−4.22	= +2.07	+1.48
34	−0.44	−0.01	+0.19	−1.60	+1.44	+0.04	−1.92	−2.50	= +1.59	+1.39
35	−0.68	−0.03	−0.51	+1.96	+1.68	+0.07	+2.58	+2.70	= +0.46	−0.68
36	−1.37	−0.06	−1.02	+2.96	+2.37	+0.10	+3.95	+3.09	= +0.71	−1.20
37	−2.43	−0.10	−2.15	+4.44	+3.42	+0.14	+5.83	+3.70	= +2.52	−0.59
38	−0.54	−0.03	+1.14	−1.33	+1.54	+0.09	−3.10	−1.20	= −0.19	−0.52
39	−2.27	−0.13	+3.71	−2.77	+3.27	+0.19	−6.49	−0.59	= +0.25	−1.20
40	−2.32	−0.13	+4.26	−2.04	+3.27	+0.19	−6.59	−0.07	= +1.34	−0.27
41	−0.46	−0.03	+1.64	−0.39	+1.46	+0.09	−3.12	+0.99	= +2.17	+1.73
42	+0.13	+0.01	−0.89	+0.28	+0.87	+0.06	+1.91	+0.52	= +1.01	+0.85
43	−0.25	−0.02	−1.34	+0.16	+1.24	+0.09	+2.71	−0.68	= −0.68	−1.18
44	−1.37	−0.10	−3.10	+0.07	+2.37	+0.17	+4.12	−2.66	= +0.01	−1.57

No.										Residuals.
45	$-2.17x$	$-0.16y$	$-4.30z$	$-1.15u$	$+3.17x'$	$+0.23y'$	$+4.76z'$	$-4.18u'$	$=+1.''79$	$-0.''31$
46	−0.86	−0.06	−2.06	−0.91	+1.86	+0.14	+2.50	−3.03	= +0.55	−0.31
47	−0.41	−0.03	−1.38	−0.72	+1.41	+0.11	+1.57	−2.63	= +1.49	+1.01
48	+0.28	+0.02	+0.81	−0.23	+0.72	+0.06	−1.45	−0.59	= −2.32	−2.18
49	+0.13	+0.01	+0.92	+0.17	+0.87	+0.08	−1.91	+0.43	= −0.21	−0.26
50	−0.05	0.00	+1.06	+0.48	+1.04	+0.09	−1.96	+1.41	= −0.98	−1.26
51	−0.80	−0.07	+1.95	+1.04	+1.79	+0.16	−2.15	+3.30	= +0.98	−0.08
52	−2.13	−0.19	+3.57	+2.60	+3.10	+0.28	−2.37	+5.96	= +4.52	+1.97
53	−2.58	−0.23	+3.49	+3.83	+3.54	+0.32	−2.09	+6.93	= +3.16	−0.06
54	−1.28	−0.12	+1.70	+2.46	+2.26	+0.20	−0.80	+4.79	= +0.13	−1.67
55	−0.48	−0.04	+0.81	+1.49	+1.47	+0.13	+0.19	+3.29	= +1.94	+1.03
56	+0.09	+0.01	−0.55	−0.78	+0.92	+0.09	+1.02	−1.78	= +0.55	+0.58
57	−0.17	−0.02	−0.62	−1.07	+1.18	+0.12	+0.55	−2.81	= +0.44	+0.27
58	−1.03	−0.11	−1.08	−2.31	+2.05	+0.21	−0.57	−4.25	= +0.95	+0.14
59	−2.27	−0.24	−1.37	−4.41	+3.29	+0.35	−1.62	−6.32	= +1.02	−0.62
60	−0.52	−0.05	+0.10	−1.72	+1.52	+0.16	−1.83	−2.76	= +2.87	+2.49
61	−0.06	−0.01	+0.16	−1.09	+1.07	+0.11	−1.89	−1.47	= +2.61	+2.58
62	+0.07	+0.01	−0.02	+1.02	+0.92	+0.11	+0.26	+2.12	= −0.58	−0.88
63	−0.12	−0.01	−0.16	+1.21	+1.11	+0.13	+1.00	+2.37	= +0.07	−0.48
64	−0.62	−0.07	−0.36	+1.89	+1.61	+0.19	+2.35	+2.74	= +0.72	−0.48
65	−2.12	−0.26	−1.60	+4.07	+3.07	+0.37	+5.13	+3.73	= +1.87	−1.35
66	−2.11	−0.26	−2.45	+3.60	+3.07	+0.37	+5.44	+3.15	= +4.21	+1.04
67	−0.23	−0.03	−0.97	+0.86	+1.22	+0.15	+2.71	+0.18	= +0.87	+0.23
68	+0.03	0.00	−0.80	+0.55	+0.96	+0.12	+2.05	−0.56	= +0.23	−0.06
69	+0.09	+0.01	+0.47	−0.82	+0.92	+0.12	−1.17	−1.65	= −0.45	−0.54
70	−0.17	−0.02	+0.73	−1.00	+1.18	+0.16	−2.14	−1.49	= +0.89	+0.73
71	−0.41	−0.05	+0.98	−1.22	+1.41	+0.19	−2.79	−1.34	= +3.22	+2.87
72	−1.19	−0.16	+1.98	−2.02	+2.20	+0.30	−4.49	−1.05	= −1.61	−2.63
73	−2.29	−0.31	+3.67	−2.89	+3.29	+0.45	−6.57	−0.78	= +0.77	−1.19
74	−1.13	−0.16	+2.56	−0.97	+2.14	+0.29	−4.55	+0.41	= +3.09	+1.95
75	−0.31	−0.04	+1.41	−0.36	+1.31	+0.18	−2.77	+1.06	= +3.64	+3.24
76	+0.13	+0.02	−0.88	+0.33	+0.97	+0.15	+1.93	+0.85	= −1.06	−1.21
77	−0.48	−0.07	−1.66	+0.23	+1.64	+0.25	+3.39	−0.84	= +0.13	−0.74
78	−1.09	−0.17	−2.64	+0.21	+2.25	+0.34	+4.25	−1.90	= +1.03	−0.58
79	−2.05	−0.32	−4.24	−0.17	+3.16	+0.48	+5.41	−3.27	= +1.30	−1.37
80	−2.51	−0.39	−4.99	−0.64	+3.50	+0.54	+5.80	−3.80	= +3.15	+0.05
81	−1.84	−0.28	−3.75	−1.02	+2.67	+0.41	+4.44	−3.24	= +3.72	+1.53
82	−0.78	−0.12	−2.03	−0.71	+1.64	+0.25	+2.66	−2.47	= +2.94	+1.95
83	−0.45	−0.07	−1.47	−0.65	+1.27	+0.20	+1.89	−2.22	= +1.89	+1.30
84	−0.07	−0.01	−0.98	−0.51	+0.94	+0.15	+0.91	−2.01	= +1.51	+1.32
85	+0.04	+0.01	+1.02	+0.20	+1.08	+0.18	−2.25	+0.55	= +0.69	+0.53
86	−0.24	−0.04	+1.28	+0.45	+1.39	+0.23	−2.49	+1.68	= +0.95	+0.41
87	−0.51	−0.09	+1.62	+0.66	+1.67	+0.28	−2.61	+2.47	= −0.10	−1.00
88	−1.36	−0.23	+2.76	+1.48	+2.52	+0.43	−2.92	+4.38	= +2.78	+0.78
89	−2.36	−0.40	+3.97	+2.75	+3.45	+0.58	−3.24	+6.23	= +5.62	+2.27
90	−1.95	−0.33	+2.75	+3.06	+2.73	+0.46	−2.12	+5.43	= +4.99	+2.07
91	−0.20	−0.03	+0.64	+1.14	+1.02	+0.17	+0.16	+2.49	= +2.58	+2.00
92	+0.24	+0.04	−0.01	+0.86	+0.66	+0.11	+1.23	+0.92	= +0.53	+0.47
93	+0.04	+0.01	−0.58	−0.82	+0.97	+0.18	+0.98	−1.93	= −1.45	−1.45
94	−0.14	−0.02	−0.64	−1.01	+1.15	+0.21	+0.65	−2.46	= +0.68	+0.49
95	−0.35	−0.06	−0.76	−1.27	+1.36	+0.25	+0.34	−2.95	= +1.31	+0.90
96	−0.81	−0.15	−1.05	−1.93	+1.83	+0.34	−0.20	−3.87	= +0.68	−0.18
97	−1.75	−0.32	−1.47	−3.45	+2.78	+0.51	−1.00	−5.52	= +2.86	+1.13

No.										Residuals.
98	−2.47x	−0.46y	−1.39z	−4.75u	+3.49x'	+0.65y'	−1.61z'	−6.68u'	= +2″.36	+0″.02
99	−2.10	−0.39	−0.73	−4.28	+3.12	+0.58	−1.72	−5.98	= +2.60	+0.64
100	−1.05	−0.20	−0.11	−2.57	+2.06	+0.38	−1.67	−3.99	= +0.79	−0.19
101	−0.60	−0.11	+0.02	−1.85	+1.61	+0.30	−1.70	−3.02	= +1.48	+0.92
102	−0.35	−0.06	+0.06	−1.47	+1.35	+0.25	−1.75	−2.40	= −0.92	−1.26
103	−0.09	−0.02	+0.11	−1.13	+1.09	+0.20	−1.82	−1.62	= +0.20	+0.12
104	+0.12	+0.02	+0.09	+0.96	+0.87	+0.17	−0.09	+2.01	= +0.46	+0.23
105	−0.09	−0.02	−0.10	+1.21	+1.10	+0.22	+0.87	+2.40	= +0.79	+0.26
106	−0.32	−0.06	−0.20	+1.48	+1.31	+0.26	+1.49	+2.60	= +1.84	+0.95
107	−1.24	−0.25	−0.70	+2.84	+2.22	+0.44	+3.45	+3.30	= +5.70	+3.37
108	−2.63	−0.53	−2.15	+4.76	+3.58	+0.72	+5.86	+4.25	= +4.49	−0.01
109	−1.44	−0.29	−1.91	+2.58	+2.42	+0.49	+4.44	+2.42	= +3.11	+0.51
110	−0.63	−0.13	−1.27	+1.40	+1.62	+0.33	+3.34	+1.13	= +2.10	+0.78
111	−0.42	−0.09	−1.09	+1.13	+1.41	+0.29	+3.02	+0.72	= +4.67	+3.68
112	−0.20	−0.04	−0.92	+0.87	+1.20	+0.24	+2.65	+0.23	= +1.74	+1.08
113	+0.14	+0.03	−0.76	+0.46	+0.86	+0.17	+1.70	−0.82	= +0.76	+0.63
114	+0.24	+0.05	−0.80	+0.21	+0.75	+0.15	+1.09	−1.20	= +1.98	+2.03
115	+0.39	+0.08	−0.49	−0.22	+0.68	+0.14	+0.24	−1.39	= +0.96	+1.27
116	+0.37	+0.08	−0.55	−0.61	+0.64	+0.13	−0.42	−1.24	= −0.50	−0.12
117	+0.38	+0.08	−0.29	−0.77	+0.62	+0.13	−0.77	−1.00	= −1.55	−1.11
118	+0.40	+0.08	+0.15	−0.82	+0.60	+0.12	−1.07	−0.59	= −1.13	−0.63
119	+0.41	+0.08	+0.33	−0.77	+0.60	+0.12	−1.16	−0.34	= −0.66	−0.17
120	+0.41	+0.09	+0.60	−0.59	+0.59	+0.12	−1.19	+0.28	= −2.75	−2.27
121	+0.42	+0.09	+0.81	−0.23	+0.58	+0.12	−1.08	+0.48	= −2.23	−1.81
122	+0.42	+0.09	+0.84	+0.12	+0.58	+0.12	−0.85	+0.80	= −0.45	−0.09
123	+0.42	+0.09	+0.43	+0.73	+0.57	+0.12	−0.12	+1.16	= −0.44	−0.21
124	+0.42	+0.09	+0.22	+0.82	+0.57	+0.12	+0.14	+1.16	= −0.25	−0.05
125	−0.62	−0.08	+1.24	−1.44	+1.63	+0.22	−3.30	−1.23	= +0.50	−0.03
126	−1.88	−0.26	+2.98	−3.12	+2.88	+0.39	−5.80	−0.87	= +0.49	−1.02
127	−1.65	−0.24	+3.29	−1.47	+2.65	+0.37	−5.52	+0.04	= +3.61	+2.02
128	−0.43	−0.06	+1.57	−0.43	+1.43	+0.20	−3.07	+0.93	= +0.82	+0.31
129	+0.18	+0.03	−0.85	+0.36	+0.81	+0.12	+1.70	+0.76	= +0.51	+0.43
130	+0.06	+0.01	−1.00	+0.24	+0.96	+0.14	+2.18	+0.27	= −0.35	−0.56
131	−0.15	−0.02	−1.22	+0.19	+1.15	+0.17	+2.58	−0.38	= +0.88	+0.42
132	−0.57	−0.09	−1.81	+0.19	+1.56	+0.24	+3.19	−1.29	= −1.10	−2.05
133	−1.62	−0.25	−3.52	+0.03	+2.67	+0.41	+4.51	−2.92	= +0.68	−1.50
134	−2.41	−0.37	−4.84	−0.46	+3.57	+0.55	+5.46	−4.03	= +3.34	+0.29
135	−2.21	−0.34	−4.40	−1.02	+3.22	+0.49	+4.97	−4.07	= +1.74	−0.93
136	−1.18	−0.18	−2.60	−0.95	+2.18	+0.34	+3.20	−3.21	= +1.44	0.00
137	−0.35	−0.05	−1.32	−0.63	+1.36	+0.21	+1.51	−2.54	= +1.09	+0.56
138	+0.02	0.00	−0.83	+0.52	+0.97	+0.15	+0.35	−2.12	= +1.32	+1.23
139	+0.15	+0.03	+0.92	+0.09	+0.84	+0.14	−1.89	+0.22	= −0.60	−0.59
140	−0.07	−0.01	+1.10	+0.35	+1.07	+0.18	−2.09	+1.26	= −0.20	−0.50
141	−0.30	−0.05	+1.35	+0.52	+1.29	+0.22	−2.13	+1.99	= +1.23	+0.64
142	−1.01	−0.17	+2.28	+1.15	+2.00	+0.34	−2.30	+3.70	= +2.00	+0.49
143	−2.22	−0.38	+3.80	+2.57	+3.21	+0.54	−2.66	+6.05	= +5.49	+2.34
144	−2.54	−0.43	+3.66	+3.57	+3.51	+0.60	−2.36	+6.78	= +7.55	+3.84
145	−0.61	−0.10	+1.02	+1.61	+1.60	+0.27	−1.65	+3.56	= +3.16	+1.98
146	+0.42	+0.03	+0.30	+0.80	+0.58	+0.05	+0.38	+1.13	= 0.00	−0.06
147	+0.41	+0.03	−0.80	−0.24	+0.59	+0.05	+1.02	−0.58	= −0.11	+0.11
148	+0.39	+0.03	−0.04	−0.83	+0.62	+0.05	+0.06	−1.26	= +1.08	+1.40
149	+0.34	+0.03	+0.56	−0.61	+0.66	+0.06	−0.81	−1.12	= −0.51	−0.25
150	−0.02	0.00	+0.38	+1.00	+1.04	+0.10	+1.03	+2.11	= +0.79	+0.40

No.										Residuals.
151	$+0.16x$	$+0.01y$	$+0.12z$	$+0.89u$	$+0.83x'$	$+0.08y'$	$+1.43z'$	$+1.18u'$	$=+3''.21$	$+3''.00$
152	+0.38	+0.04	—0.83	—0.03	+0.62	+0.06	+0.61	—1.11	=+0.34	+0.49
153	+0.40	+0.04	—0.58	—0.59	+0.60	+0.06	—0.13	—1.19	=—0.29	0.00
154	+0.42	+0.04	+0.06	+0.83	+0.59	+0.06	—0.83	—0.83	=—0.08	—0.03
155	+0.42	+0.04	+0.72	+0.43	+0.58	+0.06	—0.81	+0.85	=+1.04	+1.21
156	+0.41	+0.04	—0.25	—0.81	+0.59	+0.06	—0.10	+1.21	=+0.38	+0.75
157	+0.39	+0.04	—0.44	+0.72	+0.60	+0.06	+0.65	+1.08	=—0.61	—0.57
158	+0.35	+0.04	—0.85	+0.04	+0.65	+0.07	+1.38	+0.24	=—1.16	—1.04
159	+0.19	+0.02	—0.61	—0.65	+0.81	+0.08	+1.26	—1.29	=—0.19	—0.07
160	+0.21	+0.02	+0.35	—0.80	+0.77	+0.08	—1.76	—0.15	=+1.50	+1.69
161	+0.34	+0.04	+0.81	—0.24	+0.66	+0.07	—0.94	+1.06	=—0.53	—0.30
162	+0.34	+0.06	—0.63	+0.52	+0.66	+0.11	+1.25	—0.55	=+1.72	+1.80
163	+0.38	+0.07	—0.82	—0.05	+0.62	+0.11	+0.60	—1.10	=+0.98	+1.22
164	+0.42	+0.07	—0.02	—0.85	+0.59	+0.10	—0.75	—0.90	=—0.97	—0.50
165	+0.42	+0.07	+0.83	—0.13	+0.58	+0.10	—1.13	+0.28	=+0.52	+0.89
166	+0.41	+0.07	+0.64	+0.55	+0.58	+0.10	—0.70	+0.95	=+0.01	+0.24
167	+0.17	+0.03	+0.29	—0.85	+0.83	+0.16	—1.81	—0.43	=+1.38	+1.57
168	+0.35	+0.07	+0.82	—0.10	+0.65	+0.12	—1.16	+0.75	=+0.61	+0.90
169	+0.38	+0.07	+0.75	+0.38	+0.61	+0.12	—0.07	+1.29	=+0.85	+1.09
170	+0.42	+0.08	—0.79	+0.24	+0.58	+0.11	+1.10	—0.35	=—0.73	—0.48
171	+0.42	+0.08	—0.35	—0.76	+0.59	+0.11	+0.18	—1.15	=—0.27	+0.20
172	+0.42	+0.08	+0.27	—0.81	+0.58	+0.11	—0.50	—1.05	=—3.06	—2.55
173	+0.38	+0.07	+0.79	—0.27	+0.62	+0.12	—1.27	—0.50	=—0.26	+0.10
174	+0.31	+0.06	+0.81	+0.20	+0.64	+0.13	—1.35	+0.18	=—2.86	—2.65
175	+0.32	+0.06	+0.65	+0.55	+0.68	+0.13	—1.25	+0.80	=—0.73	—0.60

The equations derived from the latitudes η contain two more unknown quantities,

$$v = \Delta i', \quad w = \sin i' . \Delta \Omega',$$

but, in them, the variation of the solar elements will be neglected.

The mean of the Transits of 1761 and 1769 gives

$$-0.059x + 0.050y - 0.095z - 0.069u + 0.000v + 1.000w = -1''.165.$$

From this mean the indeterminate correction of the Sun's semi-diameter is nearly eliminated.

Equations of Condition.

No.							
1	$-0.01x$	$+0.00y$	$-0.01z$	$+0.00u$	$+0.61v$	$+1.24w$	$=+0''.82$
2	—0.10	+0.01	—0.21	—0.08	—0.36	+1.95	=+0.41
3	—0.12	+0.02	—0.31	—0.11	—1.09	+2.04	=—0.49
4	+0.17	—0.02	—0.41	+0.25	—2.13	+0.88	=—0.14
5	+0.20	—0.03	—0.37	+0.17	—1.60	—0.40	=—1.51
6	+0.09	—0.01	—0.14	—0.10	+0.12	—1.35	=+0.02
7	+0.20	—0.02	—0.23	—0.35	+1.17	—1.42	=+5.62
8	+0.19	—0.02	—0.30	—0.49	+2.32	—0.77	=+1.54
9	—0.14	+0.02	—0.54	—0.16	+2.42	+0.46	=+0.64
10	—0.23	+0.03	—0.54	—0.07	+1.88	+1.10	=—1.70
11	—0.18	+0.02	—0.36	—0.10	+1.05	+1.38	=—1.48

No.							
12	—0.22x	+0.02y	—0.01z	—0.58u	—2.34v	—0.09w	= —1.″49
13	+0.11	—0.01	—0.33	—0.36	—2.06	—1.55	= +1.04
14	+0.12	—0.01	+0.21	—0.24	+1.57	+1.68	= +0.34
15	—0.03	0.00	—0.02	—0.09	+0.06	+2.34	= +0.63
16	+0.02	0.00	+0.01	+0.05	—0.75	+1.69	= —0.77
17	+0.01	0.00	—0.07	+0.04	+0.27	—2.68	= +2.21
18	—0.15	+0.01	—0.12	+0.32	+1.60	—1.45	= +0.21
19	+0.01	0.00	+0.02	0.00	+0.78	+0.97	= +1.13
20	+0.10	0.00	—0.38	+0.18	—2.05	+1.36	= —0.77
21	+0.11	0.00	—0.17	—0.11	+0.33	—1.45	= +0.53
22	+0.23	—0.01	—0.28	—0.45	+1.63	—1.35	= +0.73
23	—0.23	+0.01	—0.57	—0.06	+2.13	+0.86	= —4.07
24	—0.13	0.00	—0.12	—0.25	—0.86	+1.09	= +0.95
25	—0.17	0.00	—0.07	—0.56	—2.43	—0.35	= —0.67
26	+0.07	0.00	—0.09	—0.11	+0.09	—1.54	= +0.53
27	+0.10	0.00	+0.18	—0.24	+1.52	+1.83	= —0.65
28	+0.01	0.00	0.00	+0.02	—0.62	+1.79	= +3.82
29	—0.06	0.00	+0.14	—0.05	—1.03	—1.91	= —2.56
30	0.00	0.00	—0.07	+0.07	+0.49	—2.67	= —0.52
31	—0.15	0.00	—0.15	+0.29	+1.60	—0.73	= —0.13
32	—0.10	0.00	—0.30	—0.06	—1.04	+2.13	= +0.55
33	+0.21	+0.01	—0.38	+0.27	—1.92	+0.18	= —0.52
34	+0.16	0.00	—0.30	+0.10	—1.27	—0.63	= —0.79
35	+0.14	+0.01	—0.21	—0.20	+0.59	—1.54	= —0.38
36	+0.22	+0.01	—0.27	—0.39	+1.41	—1.47	= +1.29
37	+0.16	+0.01	—0.37	—0.44	+2.39	—0.81	= —0.72
38	—0.16	—0.01	—0.12	—0.30	—0.98	+1.12	= +0.33
39	—0.18	—0.01	—0.09	—0.56	—2.43	—0.21	= +2.84
40	+0.17	+0.01	—0.42	—0.29	—1.88	—1.59	= —1.00
41	+0.06	0.00	—0.08	—0.10	+0.20	—1.44	= +0.10
42	+0.06	0.00	—0.06	—0.11	+0.31	—0.86	= +0.03
43	+0.13	+0.01	+0.04	—0.25	+1.18	—0.46	= +0.64
44	+0.18	+0.01	+0.22	—0.32	+1.78	+0.86	= —0.92
45	—0.05	0.00	—0.03	—0.14	+0.33	+2.36	= —0.04
46	+0.02	0.00	+0.01	+0.05	—0.80	+1.48	= +0.66
47	+0.05	0.00	—0.01	+0.10	—1.04	+0.90	= +0.64
48	—0.03	0.00	—0.05	—0.05	+0.03	+0.70	= +0.07
49	—0.07	—0.01	—0.02	—0.13	—0.68	+0.61	= —1.75
50	—0.09	—0.01	+0.09	—0.16	—1.15	+0.14	= —0.90
51	—0.11	—0.01	+0.19	—0.12	—1.37	—1.06	= —3.16
52	—0.03	0.00	+0.09	—0.03	—0.71	—2.35	= —4.41
53	—0.03	0.00	—0.07	+0.13	+0.84	—2.49	= —1.78
54	—0.14	—0.01	—0.10	+0.29	+1.54	—1.26	= +3.67
55	—0.12	—0.01	—0.14	+0.19	+1.45	—0.21	= +0.35
56	+0.04	0.00	+0.06	—0.05	+0.93	+0.14	= +1.39
57	+0.03	0.00	+0.05	—0.02	+0.90	+.079	= +1.57
58	—0.05	0.00	—0.10	—0.02	+0.18	+1.80	= +2.67
59	—0.08	—0.01	—0.30	—0.03	—1.11	+2.13	= +2.72
60	+0.16	+0.02	—0.31	+0.13	—1.37	—0.50	= +0.58
61	+0.10	+0.01	—0.19	—0.01	—0.63	—0.90	= —0.01
62	0.00	0.00	—0.01	0.00	—0.70	—0.73	= —1.35
63	+0.04	0.00	—0.07	—0.02	—0.37	+1.15	= —1.50
64	+0.13	+0.01	—0.20	—0.17	+0.44	—1.54	= —0.74
65	+0.22	+0.03	—0.34	—0.47	+2.04	—1.19	= —3.38

No.							
66	$-0.21x$	$-0.03y$	$-0.59z$	$-0.02u$	$+2.26v$	$+0.54w$	$= +2''.46$
67	-0.08	-0.01	-0.14	-0.09	$+0.16$	$+1.25$	$= -0.29$
68	-0.03	0.00	-0.03	-0.05	-0.40	$+0.89$	$= +3.00$
69	-0.04	0.00	-0.07	-0.03	$+0.26$	$+0.90$	$= +0.62$
70	-0.07	-0.01	-0.09	-0.11	-0.36	$+1.15$	$= -2.16$
71	-0.14	-0.02	-0.13	-0.25	-0.80	$+1.15$	$= -3.21$
72	-0.23	-0.03	-0.07	-0.49	-1.74	$+0.76$	$= -0.32$
73	-0.17	-0.02	-0.12	-0.56	-2.41	-0.21	$= +2.40$
74	$+0.16$	$+0.02$	-0.28	-0.18	-0.74	-1.75	$= -2.59$
75	$+0.04$	$+0.01$	-0.05	-0.07	$+0.33$	-1.30	$= -0.70$
76	$+0.06$	$+0.01$	-0.07	-0.10	$+0.26$	-0.86	$= -0.80$
77	$+0.16$	$+0.02$	$+0.09$	-0.30	$+1.43$	-0.21	$= +0.69$
78	$+0.19$	$+0.03$	$+0.21$	-0.34	$+1.75$	$+0.50$	$= +0.68$
79	$+0.13$	$+0.02$	$+0.20$	-0.30	$+1.72$	$+1.54$	$= -0.50$
80	$+0.01$	0.00	$+0.03$	-0.17	$+1.22$	$+2.20$	$= +0.45$
81	-0.04	-0.01	-0.03	-0.09	$+0.07$	$+2.22$	$= +0.26$
82	$+0.02$	0.00	$+0.01$	$+0.04$	-0.76	$+1.50$	$= -0.82$
83	$+0.04$	$+0.01$	0.00	$+0.08$	-0.99	$+1.02$	$= -0.72$
84	$+0.05$	$+0.01$	-0.04	$+0.10$	-1.06	$+0.30$	$= -0.07$
85	-0.08	-0.01	$+0.01$	-0.16	-0.88	$+0.49$	$= +0.83$
86	-0.11	-0.02	$+0.12$	-0.17	-1.29	-0.09	$= -1.35$
87	-0.12	-0.02	$+0.17$	-0.16	-1.41	-0.56	$= +0.45$
88	-0.10	-0.02	$+0.20$	-0.09	-1.26	-1.63	$= -0.07$
89	-0.03	0.00	$+0.07$	-0.04	-0.61	-2.49	$= +0.58$
90	-0.09	-0.02	-0.05	$+0.23$	$+1.23$	-2.00	$= +1.14$
91	-0.11	-0.02	-0.17	$+0.15$	$+1.24$	$+0.17$	$= +0.70$
92	-0.04	-0.01	-0.07	-0.03	$+0.01$	$+0.75$	$= -0.49$
93	$+0.04$	$+0.01$	$+0.06$	-0.05	$+0.96$	$+0.26$	$= +0.83$
94	$+0.03$	$+0.01$	$+0.06$	-0.02	$+0.95$	$+0.69$	$= +0.99$
95	$+0.02$	0.00	$+0.04$	-0.01	$+0.82$	$+1.06$	$= +0.89$
96	-0.02	0.00	-0.04	0.00	$+0.44$	$+1.61$	$= +0.92$
97	-0.09	-0.02	-0.21	-0.05	-0.44	$+2.14$	$= +0.44$
98	-0.03	0.00	-0.31	$+0.05$	-1.44	$+2.02$	$= +1.56$
99	$+0.14$	$+0.03$	-0.33	$+0.27$	-1.96	$+1.25$	$= -1.15$
100	$+0.20$	$+0.04$	-0.34	$+0.26$	-1.79	$+0.12$	$= -0.91$
101	$+0.17$	$+0.03$	-0.31	$+0.17$	-1.48	-0.35	$= +0.19$
102	$+0.14$	$+0.03$	-0.27	$+0.09$	-1.18	-0.62	$= -0.52$
103	$+0.10$	$+0.02$	-0.20	0.00	-0.72	-0.86	$= -0.60$
104	-0.01	0.00	$+0.01$	0.00	-0.79	-0.52	$= +0.02$
105	$+0.03$	$+0.01$	-0.06	-0.01	-0.43	-1.12	$= +0.74$
106	$+0.07$	$+0.01$	-0.13	-0.06	-0.08	-1.38	$= +1.18$
107	$+0.20$	$+0.04$	-0.28	-0.33	$+1.14$	-1.61	$= +0.54$
108	$+0.05$	$+0.01$	-0.46	-0.30	$+2.47$	-0.62	$= +0.96$
109	-0.24	-0.05	-0.53	0.00	$+1.77$	$+0.94$	$= +0.22$
110	-0.16	-0.03	-0.31	-0.07	$+0.88$	$+1.27$	$= +1.94$
111	-0.12	-0.02	-0.23	-0.09	$+0.56$	$+1.28$	$= +1.03$
112	-0.08	-0.02	-0.14	-0.09	$+0.16$	$+1.20$	$= +1.81$
113	-0.01	0.00	-0.01	-0.03	-0.57	$+0.65$	$= +0.34$
125	-0.17	-0.02	-0.12	-0.33	-1.12	$+1.08$	$= +1.69$
126	-0.23	-0.03	-0.07	-0.49	-2.24	$+0.23$	$= +1.42$
127	$+0.19$	$+0.03$	-0.39	-0.22	-1.30	-1.73	$= +0.59$
128	$+0.06$	$+0.01$	-0.08	-0.10	$+0.16$	-1.43	$= +0.76$
129	$+0.05$	$+0.01$	-0.07	-0.08	$+0.11$	-0.85	$= -0.11$
130	$+0.08$	$+0.01$	-0.05	-0.16	$+0.56$	-0.85	$= +0.64$

No.							
131	+0.12x	+0.02y	+0.01z	—0.24u	+1.03v	—0.61w	= +0″.74
132	+0.17	+0.02	+0.12	—0.31	+1.51	—0.08	= +1.60
133	+0.18	+0.03	+0.24	—0.33	+1.81	+1.08	= +1.68
134	+0.07	+0.01	+0.13	—0.29	+1.47	+1.98	= —0.50
135	—0.06	—0.01	—0.05	—0.18	+0.46	+2.34	= +0.90
136	—0.01	0.00	0.00	—0.01	—0.51	+1.80	= —0.11
137	+0.05	+0.01	—0.01	+0.09	—1.03	+0.84	= —0.35
138	+0.05	+0.01	—0.06	+0.09	—0.99	+0.03	= +0.88
139	—0.06	—0.01	—0.02	—0.12	—0.56	+0.66	= —0.91
140	—0.09	—0.02	+0.07	—0.17	—1.13	+0.23	= —0.84
141	—0.11	—0.02	+0.14	—0.17	—1.34	—0.22	= —1.90
142	—0.11	—0.02	+0.21	—0.11	—1.37	—1.26	= —1.37
143	—0.04	—0.01	+0.11	—0.05	—0.77	—2.38	= —0.65
144	—0.03	—0.01	—0.05	+0.12	+0.77	—2.51	= +3.52
145	—0.14	—0.02	—0.14	+0.24	+1.48	—0.50	= +0.28

To apply to these equations the rigorous method of least squares would be very laborious; hence the method of "Equivalent Factors" has been used; the equations have been multiplied either by whole numbers or by fractions which are ready multipliers. In this way the following *Normal Equations* were derived from the equations of condition which have $\cos\eta . \Delta\theta$ for their absolute terms:

+195.84x	—44.809y	+127.71z	+ 73.19u	—251.90x'	+43.027y'	— 85.48z'	+119.25u'	= — 8″.77
— 44.78	+47.099	— 83.68	— 62.84	+ 41.04	—48.460	+ 41.17	— 96.06	= —113.43
+120.94	—83.889	+427.28	+133.17	—136.59	+82.936	—410.76	+400.15	= +162.30
+ 70.03	—62.965	+135.64	+365.81	— 73.13	+63.350	+114.76	+508.04	= +197.06
—255.15	+42.172	—138.12	— 80.06	+425.64	—27.182	+ 91.22	—132.67	= + 92.63
+ 40.68	—48.373	+ 82.84	+ 61.99	— 26.27	+51.815	— 41.45	+ 94.13	= +121.18
— 83.42	+41.537	—422.53	+119.76	+102.83	—40.091	+644.06	—111.82	= — 23.87
+112.81	—95.792	+406.68	+505.65	—126.69	+94.621	—120.34	+902.21	= +264.18

If u is eliminated from these equations, the result is

+181.83x	—32.213y	+100.57z	—237.27x'	+30.352y'	—108.44z'	+ 17.60u'	= — 48″.20
— 32.75	+36.284	— 60.38	+ 28.48	—37.577	+ 60.88	— 8.78	= — 79.58
+ 95.45	—60.971	+377.90	—109.97	+59.874	—452.54	+215.20	= + 90.56
—239.82	+28.394	—108.43	+409.63	—13.317	+116.34	— 21.48	= +135.76
+ 28.81	—37.705	+ 59.85	— 13.88	+41.080	— 60.90	+ 8.04	= + 87.79
—106.35	+62.147	—466.94	+126.77	—60.831	+606.49	—278.15	= — 88.38
+ 16.01	— 8.770	+219.18	— 25.60	+ 7.053	—278.97	+199.94	= — 8.21

and if from these z is eliminated, the result is

+156.43x	—15.987y	—208.00x'	+14.418y'	+11.99z'	—39.67u'	= — 72″.30
— 17.50	+26.542	+ 10.91	—28.055	—11.42	+25.60	= — 65.11
—212.43	+10.900	+378.08	+ 3.863	—13.51	+40.27	= +161.74
+ 13.69	—28.049	+ 3.54	+31.598	+10.77	—26.04	= + 73.45
+ 11.59	—13.190	— 9.11	+13.151	+47.33	—12.25	= + 23.52
— 39.35	+26.593	+ 38.18	—27.674	—16.50	+75.13	= — 61.46

It is evident now, that since the principal coefficients of z' and u' have fallen from 644.06 and 902.21 to 47.33 and 75.13, no very reliable values of these quantities can be obtained from these equations. The elimination of y gives

$+145.89x$	$-201.43x'$	$-\ 2.480y'$	$+\ 5.11z'$	$-24.25u'$	$=-111''.52$
-205.24	$+373.60$	$+15.384$	$-\ 8.82$	$+29.76$	$=+188.48$
$-\ \ 4.80$	$-\ 15.07$	$+\ 1.950$	$-\ 1.30$	$-\ 1.01$	$=+\ \ 4.64$
$+\ \ 2.89$	$-\ \ 3.69$	$-\ 0.791$	$+41.65$	$+\ 0.47$	$=-\ \ 8.84$
$-\ 21.82$	$+\ 27.25$	$+\ 0.435$	$-\ 5.06$	$+49.48$	$=+\ \ 3.78$

The elimination of x from these gives

$+90.23x'$	$+11.895y'$	$-\ 1.63z'$	$-\ 4.35u'$	$=+31''.63$
$+\ 8.44$	$+\ 1.868$	$-\ 1.13$	$+\ 0.21$	$=+\ 0.97$
$+\ 0.30$	$-\ 0.742$	$+41.55$	$+\ 0.95$	$=-\ 6.63$
$-\ 2.88$	$+\ 0.064$	$-\ 4.30$	$+45.85$	$=-12.89$

The elimination of x' from these gives

$+0.755y'$	$-\ 0.98z'$	$+\ 0.62u'$	$=-\ 1''.99$
-0.782	$+41.56$	$+\ 0.96$	$=-\ 6.74$
$+0.444$	$-\ 4.35$	$+45.71$	$=-11.88$

The only condition, relative to the solar elements, which can be obtained with any weight from these equations is

$$x' + 0.132y' = +\,0''.335\,.$$

That is, the mean longitude of the Sun of Hansen and Olufsen's Tables ought to be increased by a third of a second at the epoch 1863. As, however, these Tables will, probably, be used for a long time to come in computing the solar coordinates of the *American Ephemeris*, y', z' and u' will be put severally equal to zero; and, as it has been decided to use the Pulkova constant of aberration, x' will be put equal to $+0''.19$. With these assumptions, the values of x, y, z and u are

$$x = -\,0''.502\,,\quad y = -\,2''.863\,,\quad z = -\,0''.040\,,\quad u = +\,0''.195\,.$$

The equation of condition derived from the Transits of 1761 and 1769 being excluded, the normal equations, determining the corrections of the inclination and the longitude of the ascending node, are

$+2.51x$	$+0.390y$	$+1.84z$	$-0.67u$	$+163.26v$	$-\ \ 0.42w$	$=+26''.02$
-4.46	-0.105	-0.29	-1.06	$-\ \ 5.86$	$+188.58$	$=+24.11$

From these are obtained the following values of v and w:

$$v = +\,0''.18\,,\quad w = +\,0''.12 \text{ or } \Delta\Omega' = +\,2''.0\,.$$

But, from the equation furnished by the Transits in 1761 and 1769,

$$\Delta\Omega' = -17''.84.$$

If the first result is supposed to belong to 1855.0, and the second to 1765.4 the proper value of the correction is

$$\Delta\Omega' = +0''.9 + 0''.222t.$$

The origin of the pretty large correction $-0''.02863$, of the mean motion of Venus, is easily shown. In his investigation, Leverrier (*Annales*, Vol. VI, p. 72) found the following value of $\Delta n'$:

$$\Delta n' = +0''.00035 + 0''.0689\nu + 0''.0959\nu' + 0''.1207\nu'';$$

but the value of this quantity used in forming his Tables is the first term only. If the values of ν, ν', ν'' corresponding to the change from Leverrier's values of the masses to those here adopted, be substituted in this expression, the correction of Leverrier's mean motion, from this cause, is found to be

$$\Delta n' = -0''.01588.$$

Moreover, a comparison of the values of the Sun's mean longitude in the Tables of Hansen and Olufsen and of Leverrier gives

$$\text{Han.-Lev.} = -0''.93 - 0''.01074t.$$

From the way in which $\Delta n'$ and $\Delta n''$ are involved in the equations of condition, it may be concluded, that if $\Delta n''$ were left indeterminate in the solution, the value of $\Delta n'$, obtained, would be roughly

$$\Delta n' = (\Delta n') + 1.2\Delta n'',$$

$(\Delta n')$ denoting the value of $\Delta n'$ on the supposition of $\Delta n'' = 0$. Thus, on making $\Delta n'' = -0''.01074$, the correction of the mean motion of Venus from this cause is $\Delta n' = -0''.01289$. The sum of these two corrections is $\Delta n' = -0''.02877$, which is almost identical with that derived from the equations of condition.

The increment of the motion of the node, $0''.222$, requires that the mass of Venus should be reduced from $\frac{1}{408134}$ to $\frac{1}{427240}$. This agrees with Leverrier's result: setting out with the mass 0.0000024885, he found that it should be multiplied by the factor 0.948, which would make the mass $\frac{1}{423900}$.

The corrections to be added to the elements, with which we set out, to obtain the elements, from which the Tables are constructed, are

$$\Delta L' = -0''.502, \quad \Delta\pi' = +28''.46, \quad \Delta\Omega' = +0''.90 + 0''.222t,$$
$$\Delta i' = +0''.18, \quad \Delta e' = -0.000000196, \quad \Delta n' = -0''.02863.$$

The Tables have been compared with the occultation of Mercury by Venus, observed at Greenwich, May 28, 1737. The observations made are

Greenwich M. T.	
$9^h\ 40^m\ 3^s.9$.	Mercury distant from Venus not more than a tenth part of the diameter of Venus.
9 48 10.2.	Mercury wholly occulted by Venus.

The position of Mercury being derived from Prof. Winlock's Tables, the apparent position of the two planets, as seen from Greenwich, and in longitude and latitude, are

Greenwich M. T.	l	b	l'	b'	$l'-l$	$b'-b$
1737 d h May 28 8	89° 24′ 23″.05	+2° 9′ 12″.90	89° 31′ 49″.97	+2° 10′ 9″.98	+446″.92	+57″.08
9	89 27 56.68	+2 9 5.67	89 31 14.38	+2 9 42.02	+197.70	+36.35
10	89 31 30.35	+2 8 58.43	89 30 39.63	+2 9 14.28	— 50.72	+15.85

and, interpolating,

Greenwich M. T.	$l'-l$	$b'-b$	Dist. of Centers.
h m s 9 40 3.9	+31″.73	+22″.64	38″.96
9 48 10.2	— 1.79	+19.87	19.95

With the addition of 0″.57 for irradiation, the semi-diameters of Mercury and Venus are respectively 3″.98 and 26″.97; hence, at the first observation, the distance of the limbs of the planets is 8″.01, 2″.6 more than a tenth part of the diameter of Venus; at the second observation, the distance of the centers is less than the difference of the semi-diameters; hence, the Tables are verified by the statement of the observer. Venus being, at the time, a thin crescent, and about half of Mercury's disc being illuminated, it is plain that it would be difficult for the observer to estimate the distance in fractional parts of the apparent diameter of Venus.

Leverrier's remarks on this occultation are impaired by a mistake made in the last line of his computation.

MEMOIR No. 11.

On the Derivation of the Mass of Jupiter from the Motion of certain Asteroids.

(Memoirs of the American Academy of Arts and Sciences, Vol. IX, New Series, pp. 417-420, 1873.)

The object of the present note is to show that the discussion of the observations of certain asteroids, provided they extend over a sufficient period of time, will furnish a far more accurate value of the mass of Jupiter than can be obtained from measurements of the elongation of the satellites, or from the Jupiter perturbations of Saturn. It is to be hoped that observers will hereafter pay particular attention to those asteroids which are best adapted for the end in question.

The magnitude of the Jupiter perturbations of an asteroid depends at once on the magnitude of the least distance of the two bodies, and the greater or less degree of approach to commensurability of the ratio of their mean motions, and also on the magnitude of the eccentricity of the asteroid's orbit.

Those asteroids which lie on the outer edge of the group, and whose mean motions are nearly double that of Jupiter, will best fulfil the two first conditions named above. For they will have inequalities of long period whose coefficients will be of the order of the first power only of the eccentricities, while all other classes of long-period inequalities are necessarily of higher orders, and hence demand longer periods in order to have their coefficients brought up to an equal magnitude.

In order to exhibit the relative value of these asteroids for the purpose in view, I have computed the terms of the lowest order in the coefficients of these inequalities of long period for all the asteroids, yet discovered, whose daily mean motion lies between the limits 550″ and 650″; and have appended herewith tables, by which the value of these terms can be readily computed for any which may hereafter be discovered between these limits.

The formulas for computing these terms are found in the *Mécanique Céleste*, Tom. I, pp. 279–281. Here i must be put equal to 2, in the terms which involve the simple power of the eccentricities. We will employ the usual notation for the designation of the elements of orbits, and make some reductions in Laplace's formulas for the sake of ready computation.

If we put $\gamma = \frac{2\mu' - \mu}{\mu}$ or in Laplace's notation $\frac{2n' - n}{n}$, and recollect that we need the formulas only for the case of an inferior perturbed by a superior planet; and moreover make

$$\gamma F^{(2)} = -H, \quad \text{and} \quad \gamma G^{(2)} = J,$$

$F^{(2)}$ and $G^{(2)}$ being Laplace's symbols, we shall have

$$H = \frac{1}{1-\gamma^2}\left\{\frac{2\gamma(7-\gamma^2)}{1-\gamma}\alpha b_{\frac{1}{2}}^{(2)} + \frac{3 - 11\gamma + 3\gamma^2 - \gamma^3}{2-\gamma}\left[\alpha^2\frac{db_{\frac{1}{2}}^{(2)}}{d\alpha} + \frac{4}{1-\gamma}\alpha b_{\frac{1}{2}}^{(2)}\right] - \gamma\alpha^3\frac{d^2 b_{\frac{1}{2}}^{(2)}}{d\alpha^2}\right\},$$

$$J = \frac{\alpha^2}{2(1-\gamma^2)}\left\{(3+\gamma^2)\left[3\frac{b_{\frac{1}{2}}^{(1)}}{\alpha} + \frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} - 4\right] - 8\gamma\left[\frac{db_{\frac{1}{2}}^{(1)}}{d\alpha} + \tfrac{1}{4}\alpha\frac{d^2 b_{\frac{1}{2}}^{(1)}}{d\alpha} - 1\right]\right\}.$$

If, in the next place, K and β are derived from the equations

$$K\cos(\beta - \pi) = H\sin\varphi - J\sin\varphi'\cos(\pi' - \pi),$$
$$K\sin(\beta - \pi) = \quad - J\sin\varphi'\sin(\pi' - \pi),$$

the inequality in longitude we are computing is

$$\frac{m'}{\gamma^2}K\sin[L - 2L' + \beta].$$

H and J may be regarded as functions of α, and are positive between the limits corresponding to $\mu = 550''$ and $\mu = 650''$. The common logarithms of these quantities are here tabulated for every 0.001 of α between the limits above mentioned; the values of $b_{\frac{1}{2}}^{(1)}$ and $b_{\frac{1}{2}}^{(2)}$ and their differentials were obtained from Runkle's *Tables of the Coefficients of the Perturbative Function.*

α	log H	log J	α	log H	log J
0.595	0.3153369	9.871828	0.610	0.3323864	9.889836
.596	.3165277	.873131	.611	.3334562	.890910
.597	.3177113	.874420	.612	.3345169	.891967
.598	.3188875	.875695	.613	.3355683	.893007
.599	.3200561	.876956	.614	.3366103	.894030
.600	.3212173	.878202	.615	.3376427	.895036
.601	.3223707	.879434	.616	.3386652	.896022
.602	.3235163	.880652	.617	.3396777	.896990
.603	.3246540	.881855	.618	.3406801	.897939
.604	.3257838	.883043	.619	.3416723	.898869
.605	.3269054	.884214	.620	.3426539	.899780
.606	.3280187	.885370	.621	.3436248	.900671
.607	.3291236	.886511	.622	.3445848	.901542
.608	.3302199	.887636	.623	.3455337	.902392
0.609	0.3313075	9.888745	0.624	0.3464714	9.903221

a	log H	log J	a	log H	log J
0.625	0.3473975	9.904028	0.643	0.3618323	9.914446
.626	.3483119	.904814	.644	.3624928	.914764
.627	.3492144	.905578	.645	.3631366	.915051
.628	.3501047	.906320	.646	.3637632	.915306
.629	.3509827	.907040	.647	.3643722	.915528
.630	.3518480	.907736	.648	.3649632	.915717
.631	.3527005	.908408	.649	.3655358	.915871
.632	.3535399	.909056	.650	.3660897	.915991
.633	.3543659	.909679	.651	.3666246	.916076
.634	.3551782	.910277	.652	.3671400	.916125
.635	.3559767	.910850	.653	.3676354	.916136
.636	.3567612	.911396	.654	.3681103	.916108
.637	.3575313	.911916	.655	.3685644	.916040
.638	.3582866	.912409	.656	.3689972	.915933
.639	.3590269	.912874	.657	.3694082	.915785
.640	.3597519	.913310	.658	.3697969	.915595
.641	.3604614	.913718	.659	.3701628	.915362
0.642	0.3611550	9.914097	0.660	0.3705053	9.915085

The values of the elements of Jupiter's orbit for the epoch 1850.0 which we shall use are

$$\begin{aligned} m' &= \tfrac{1}{1050}, \\ \mu' &= 299.1286, \\ \log a' &= 0.7162372, \\ \varphi' &= 2^\circ\, 45'\, 54''.55, \\ \pi' &= 11^\circ\, 55'\, 2''. \end{aligned}$$

The values of the corresponding elements of as many of the asteroids as lie between the limits above mentioned are contained in the following table. The longitudes of the perihelia are referred to the mean equinox of 1850.0.

	μ	log a	ϕ	π
Hygea	634″.3118	0.498 4692	5° 44′ 56″.4	234° 58′ 40″.6
Themis	636.7634	0.497 3523	6 42 52.9	139 56 11.2
Euphrosyne	633.8508	0.498 8680	12 44 10.3	93 27 51.5
Doris	647.1295	0.492 6769	4 23 42.9	74 10 11.3
Pales	655.6209	0.488 9025	13 43 18.3	32 3 13.1
Europa	650.0877	0.491 3564	5 49 14.3	101 45 37.6
Mnemosyne	632.6897	0.499 2106	5 58 17.1	52 58 47.8
Erato	640.8591	0.495 4961	9 46 4.3	33 55 38.0
Cybele	560.8775	0.534 0920	6 54 36.4	258 11 24.3
Freia	569.0505	0.529 9038	10 49 12.0	93 2 36.6
Semele	652.9848	0.490 0690	11 49 36.5	28 25 39.1
Sylvia	543.5800	0.543 1620	4 39 22.6	337 8 6.1
Antiope	632.3591	0.499 3618	11 39 2.7	293 49 3.5

The expression of the inequalities, and the length of their periods which result from the substitution of these values of the elements in the formulas, are

Hygea	$14676''.2 \sin [L - 2L' + 228^\circ 58' 1''.4]$,	97.96 years.
Themis	$14606.2 \sin [L - 2L' + 146\ 4\ 4.5]$,	91.72 "
Euphrosyne	$28996.5 \sin [L - 2L' + 97\ 58\ 58.4]$,	99.23 "
Doris	$5086.7 \sin [L - 2L' + 85\ 41\ 49.4]$,	72.27 "
Pales	$11639.2 \sin [L - 2L' + 33\ 36\ 12.6]$,	61.57 "
Europa	$6584.4 \sin [L - 2L' + 111\ 29\ 19.2]$,	68.14 "
Mnemosyne	$12956.0 \sin [L - 2L' + 60\ 9\ 1.9]$,	102.58 "
Erato	$13654.9 \sin [L - 2L' + 36\ 21\ 16.9]$,	82.91 "
Cybele	$13145.4 \sin [L - 2L' + 251\ 13\ 31.6]$,	94.49 "
Freia	$32243.5 \sin [L - 2L' + 98\ 15\ 25.5]$,	120.93 "
Semele	$10860.7 \sin [L - 2L' + 29\ 55\ 45.1]$,	64.54 "
Antiope	$28567.8 \sin [L - 2L' + 288\ 44\ 31.6]$,	103.57 "

These expressions can be regarded as rough approximations only to the actual values of these inequalities, since all terms of the third and higher orders with respect to the eccentricities and inclinations, and of the second and higher orders with respect to the disturbing masses, have been neglected. Yet they are sufficiently exact to show the order of magnitude of the Jupiter perturbations of the asteroids in question.

The effect of these inequalities at the time of opposition will be magnified in the proportion roughly of a to $a - 1$. Thus in the case of Freia, the determination of the mass of Jupiter will depend on the observation of an arc of $12^\circ.7$.

MEMOIR No. 12.

On the Inequality of Long Period in the Longitude of Saturn, whose Argument is Six Times the Mean Anomaly of Saturn Minus Twice that of Jupiter Minus Three Times that of Uranus.

(Astronomische Nachrichten, Vol. 82, pp. 83-88, 1873.)

This inequality is proportional to the product of the masses of Jupiter and Uranus. In its coefficient we shall have regard only to the part which is divided by the square of the motion of the argument.

Employing the notation in general use, the quantities having no accent, or one, or two, according as they belong to Jupiter, Saturn or Uranus, ρ designating $\int ndt$, and putting

$$R = \frac{m}{1+m'}\left[\frac{1}{\Delta} - \frac{r'\cos\psi}{r^2}\right] + \frac{m''}{1+m'}\left[\frac{1}{\Delta''} - \frac{r'\cos\psi''}{r''^2}\right],$$

we have the well known equation

$$\frac{d^2\rho'}{dt^2} = -3a'n'd'R.$$

The symbol d' denotes differentiation with respect to the time only inasmuch as it is introduced into R by the coordinates of Saturn.

Having regard only to the perturbations which are of two dimensions with respect to the planetary masses, this equation may be written

$$\frac{d^2\delta\rho'}{dt^2} = -3a'n'd'\delta R + 3a'^2n'd'R\int d'R.$$

Let $n\delta z$ and δlr denote respectively the perturbations of the mean anomaly and of the natural logarithm of the radius vector in Hansen's method, and let the subscripts $({\scriptstyle 0})$ and $({\scriptstyle 2})$ denote the parts of any quantity which arise from the actions respectively of Jupiter and Uranus. Then R_0 being expressed as a function of the mean anomalies g and g', and R_2 as a function of g' and g'', neglecting the terms arising from the perturbations of the latitudes, since they have as factors the squares of the mutual inclinations of

the orbits, and preserving only the terms multiplied by the product of the masses of Jupiter and Uranus, we have

$$d'R\int d'R = d'R_0\int d'R_2 + d'R_2\int d'R_0\,,$$

$$\delta R = \frac{\partial R_0}{\partial g'}(n'\delta z')_2 + \frac{\partial R_2}{\partial g'}(n'\delta z')_0 + \frac{\partial R_0}{\partial g}(n\delta z)_2 + \frac{\partial R_2}{\partial g''}(n''\delta z'')_0$$
$$+ r'\frac{\partial R_0}{\partial r'}(\delta lr')_2 + r'\frac{\partial R_2}{\partial r'}(\delta lr')_0 + r\frac{\partial R_0}{\partial r}(\delta lr)_2 + r''\frac{\partial R_2}{\partial r''}(\delta lr'')_0\,.$$

Suppose now that $d'R_0$ has a term $A\sin(ig' - 2g + \varkappa)$, and that $d'R_2$ has a term $B\sin(i'g' - 3g'' + \lambda)$, where i and i' are positive or negative integers; and it is evident that terms in $d'R\int d'R$ having the argument $6g' - 2g - 3g''$ can arise only from the multiplication of such terms as these; then, having regard only to the term arising from the addition of the arguments,

$$d'R_0\int d'R_2 + d'R_2\int d'R_0 = -\tfrac{1}{2}\,\frac{(i+i')n' - 2n - 3n''}{(in' - 2n)(i'n' - 3n'')}\,AB\sin[(i+i')g' - 2g - 3g'' + \varkappa + \lambda].$$

But since $i + i' = 6$, the term in $\delta\rho'$ will have only the simple power of $6n' - 2n - 3n''$ as divisor; hence these terms will be neglected.

Moreover, if d'' denote differentiation with respect to the time inasmuch as it is introduced into δR by the coordinates of Jupiter and Uranus,

$$d'\delta R = d\delta R - d''\delta R.$$

But terms arising from $d\delta R$ are divided by the first power only of $6n' - 2n - 3n''$, hence, in any term of $\frac{d^2\delta\rho'}{dt^2}$, we may substitute $-d''\delta R$ for $d'\delta R$. Thus we obtain

$$\frac{d^2\delta\rho'}{dt^2} = 3a'n'\Big[n\frac{\partial^2 R_0}{\partial g\partial g'}(n'\delta z')_2 + n''\frac{\partial^2 R_2}{\partial g'\partial g''}(n'\delta z')_0 - n'\frac{\partial^2 R_0}{\partial g\partial g'}(n\delta z)_2 - n'\frac{\partial^2 R_2}{\partial g'\partial g''}(n''\delta z'')_0$$
$$+ n\frac{\partial\left(r'\frac{\partial R_0}{\partial r}\right)}{\partial g}(\delta lr')_2 + n''\frac{\partial\left(r'\frac{\partial R_2}{\partial r'}\right)}{\partial g''}(\delta lr')_0 - n'\frac{d\left(r\frac{\partial R_0}{\partial r}\right)}{\partial g'}(\delta lr)_2 - n'\frac{\partial\left(r''\frac{\partial R_2}{\partial r''}\right)}{\partial g'}(\delta lr'')_0\Big].$$

It is evident that, in the terms of this equation which are due to the mutual action of Jupiter and Uranus on each other, the argument $6g' - 2g - 3g''$ can only result from the addition of arguments like these:

In R_0 or $r\dfrac{\partial R_0}{\partial r}$;		in $(n\delta z)_2$ or $(\delta lr)_2$;	in R_2 or $r''\dfrac{\partial R_2}{\partial r''}$;		in $(n''\delta z'')_0$ or $(\delta lr'')_0$;
. . .		. . .	. . .		. . .
$6g'-3g$	and	$g-3g''$	$6g'-4g''$	and	$g''-2g$
$6g'-4g$	"	$2g-3g''$	$6g'-5g''$	"	$2g''-2g$
$6g'-5g$	"	$3g-3g''$	$6g'-6g''$	"	$3g''-3g$
. . .		. . .	. . .		. . .

But the coefficients of the terms in R_0 and $r\dfrac{\partial R}{\partial r}$, and in R_2 and $r''\dfrac{\partial R_2}{\partial r''}$, having the arguments of the first and third column, are quite small on account of the high multiple 6 of g'; and the perturbations of Jupiter by Uranus, having the arguments of the second column, are also small, as are also the perturbations of Uranus by Jupiter having the arguments of the fourth column. Hence it has been thought permissible to neglect these terms. Thus the formula used for the computation of this inequality is

$$\frac{d^2\delta\rho'}{dt^2} = 3a'nn'\left[\frac{\partial^2 R_0}{\partial g\partial g'}(n'\delta z')_2 + \frac{\partial\left(r'\dfrac{\partial R_0}{\partial r'}\right)}{\partial g}(\delta lr')_2\right]$$

$$+\,3a'n'n''\left[\frac{\partial^2 R_2}{\partial g'\partial g''}(n'\delta z')_0 + \frac{\partial\left(r'\dfrac{\partial R_2}{\partial r'}\right)}{\partial g''}(\delta lr')_0\right].$$

Here the argument $6g'-2g-3g''$ is produced only by the addition of arguments such as

. . .		. . .
$5g'-2g$	and	$g'-3g''$
$4g'-2g$	"	$2g'-3g''$
$3g'-2g$	"	$3g'-3g''$
. . .		. . .

belonging to terms of the several factors involved in the expression.

The values of the factors proportional to Jupiter's action on Saturn have been derived from Hansen's *Untersuchung über die gegenseitigen Störungen des Jupiters und Saturns;* the values of those proportional to the action of Uranus have been specially computed. The values of the masses adopted are $m=\frac{1}{1050}$, $m'=\frac{1}{3500}$, $m''=\frac{1}{21000}$. In the following expressions the common logarithms are written in place of the coefficients, and the values of $n'\delta z'$ and $\delta lr'$ are in seconds of arc.

$$\begin{aligned}\frac{a'(1+m')}{m}\frac{\partial^2 R_0}{\partial g\partial g'} = &+8.00004\cos(5g'-2g+66^\circ\,51'.6)\\ &+8.80451\cos(4g'-2g+97\;\;39\;.5)\\ &+9.50716\cos(3g'-2g+127\;\;52\;.9)\\ &+0.00652\cos(2g'-2g+158\;\;3\;.7)\\ &+9.79703\cos(g'-2g+258\;\;50\;.7),\end{aligned}\qquad \begin{aligned}(n'\delta z')_2 = &+1.47638\sin(g'-3g''+216^\circ\,12'.2)\\ &+1.40046\sin(2g'-3g''+60\;\;7\;.5)\\ &+0.17010\sin(3g'-3g''+112\;\;48\;.0)\\ &+8.6777\sin(4g'-3g''+129\qquad)\\ &+7.1505\sin(5g'-3g''+135\qquad),\end{aligned}$$

$$\begin{aligned}
\frac{a'(1+m')}{m}\frac{\partial\left(r'\frac{\partial R_0}{\partial r'}\right)}{\partial g} = &+7.96411\sin(5g'-2g+228^\circ\,48'.2)\\
&+8.80930\sin(4g'-2g+264\ \ 16\ .2)\\
&+9.59413\sin(3g'-2g+300\ \ 16\ .3)\\
&+0.23412\sin(2g'-2g+335\ \ 43\ .0)\\
&+9.82255\sin(\ g'-2g+256\ \ 31\ .4),
\end{aligned}
\qquad
\begin{aligned}
(\delta lr')_2 = &+0.1270\ \ \cos(\ g'-3g''+220^\circ\,58'\ )\\
&+1.08420\cos(2g'-3g''+240\ \ 20\ .9)\\
&+0.05350\cos(3g'-3g''+289\ \ 25\ \ )\\
&+8.7412\ \ \cos(4g'-3g''+302\ \ 36\ \ )\\
&+7.4637\ \ \cos(5g'-3g''+301\ \ \ \ ),
\end{aligned}$$

$$\begin{aligned}
a'\frac{\partial^2 R_2}{\partial g'\partial g''} = &+93.60517\cos(\ g'-3g''+\ 35^\circ\,16'.9)\\
&+94.72630\cos(2g'-3g''+238\ \ 47\ .0)\\
&+95.25666\cos(3g'-3g''+125\ \ 24\ .6)\\
&+94.26848\cos(4g'-3g''+146\ \ 30\ .5)\\
&+93.1858\ \ \cos(5g'-3g''+148\ \ 34\ \ ),
\end{aligned}
\qquad
\begin{aligned}
(n'\delta z')_0 = &+3.46348\sin(5g'-2g+246^\circ\,53'.9)\\
&+2.83450\sin(4g'-2g+277\ \ 10\ .8)\\
&+1.41347\sin(3g'-2g+135\ \ 15\ .1)\\
&+1.50550\sin(2g'-2g+156\ \ 17\ .9)\\
&+0.4363\ \ \sin(\ g'-2g+250\ \ 30\ \ ),
\end{aligned}$$

$$\begin{aligned}
a'\frac{\partial\left(r'\frac{\partial R_2}{\partial r'}\right)}{\partial g''} = &+93.98809\sin(\ g'-3g''+\ 40^\circ\ \ 2'.0)\\
&+94.85117\sin(2g'-3g''+250\ \ 27\ .4)\\
&+95.29557\sin(3g'-3g''+124\ \ 50\ .1)\\
&+94.1683\ \ \sin(4g'-3g''+153\ \ 26\ \ )\\
&+92.9164\ \ \sin(5g'-3g''+130\ \ 42\ \ ),
\end{aligned}
\qquad
\begin{aligned}
(\delta lr')_0 = &+1.72766\cos(5g'-2g+\ 62^\circ\,43'.0)\\
&+2.52156\cos(4g'-2g+277\ \ \ 2\ .4)\\
&+1.30103\cos(3g'-2g+141\ \ 32\ .0)\\
&+1.48499\cos(2g'-2g+156\ \ \ 1\ .7)\\
&+0.4146\ \ \cos(\ g'-2g+\ 97\ \ 54\ \ ).
\end{aligned}$$

In the next place

$$\log\frac{3\,mnn'}{2(1+m')(6n'-2n-3n'')^2} = 1.01570, \quad \log\frac{3n'n''}{2(6n'-2n-3n'')^2} = 3.18681\,.$$

Thus we get, the coefficients still replaced by their logarithms,

$$\begin{array}{rlllrl}
\delta\rho' = & +0.50212\sin(6g'-2g-3g''+103^\circ\ \ 3'.8) & + & 0.25546\sin(6g'-2g-3g''+102^\circ\,10'.8) \\
& +1.22067\sin(\quad \text{"} \quad +337\ \ 47\ .0) & + & 0.74761\sin(\quad \text{"} \quad +335\ \ 57\ .8) \\
& +0.69296\sin(\quad \text{"} \quad +\ 60\ \ 41\ .0) & + & 9.8569\ \ \sin(\quad \text{"} \quad +\ 80\ \ 40\ \ ) \\
& +9.6999\ \ \sin(\quad \text{"} \quad +107\ \ \ \ \ \ ) & + & 8.9608\ \ \sin(\quad \text{"} \quad +123\ \ \ \ \ \ ) \\
& +7.9632\ \ \sin(\quad \text{"} \quad +214\ \ \ \ \ \ ) & + & 6.8089\ \ \sin(\quad \text{"} \quad +219\ \ \ \ \ \ ) \\
& +9.1068\ \ \sin(\quad \text{"} \quad +269\ \ 46\ \ ) & + & 8.9026\ \ \sin(\quad \text{"} \quad +282\ \ 45\ \ ) \\
& +0.90920\sin(\quad \text{"} \quad +324\ \ 37\ .1) & + & 0.55954\sin(\quad \text{"} \quad +347\ \ 29\ .8) \\
& +0.66333\sin(\quad \text{"} \quad +\ 49\ \ 41\ .3) & + & 9.7834\ \ \sin(\quad \text{"} \quad +\ 86\ \ 22\ \ ) \\
& +9.9910\ \ \sin(\quad \text{"} \quad +\ 98\ \ 19\ \ ) & + & 8.8401\ \ \sin(\quad \text{"} \quad +129\ \ 28\ \ ) \\
& +8.3019\ \ \sin(\quad \text{"} \quad +\ 17\ \ 30\ \ ) & + & 6.5178\ \ \sin(\quad \text{"} \quad +\ 48\ \ 36\ \ ).
\end{array}$$

By the addition of these terms is obtained

$$\begin{aligned}
\delta\rho' &= +\,34''.752\sin(6g'-2g-3g'') + 1''.312\cos(6g'-2g-3g'') \\
&= +\,34''.776\sin(6g'-2g-3g''+2^\circ\,9'\,43'').
\end{aligned}$$

The inequality in the mean longitude of Uranus, having the same argument, has been calculated by Leverrier (*Additions aux Connaissance des Temps*, 1849, p. 85). He found

$$\delta\rho'' = +\,32''.74\sin(6g'-2g-3g''+181^\circ\,1'\,58'').$$

Thus, contrary to what might be expected, the inequality in the case of Saturn is larger than in the case of Uranus.

MEMOIR No. 13.

Charts and Tables for Facilitating Predictions of the Several Phases of the Transit of Venus in December, 1874.

(Papers relating to the Transit of Venus in 1874, Part II, 1872.)

CONTENTS.

CHARTS.

All the constants and elements which have been used in the computations on the transit are given below. The quantities having no terms multiplied by t are either constant or may be regarded as such for the duration of the transit; and the quantities which vary may be regarded as varying uniformly. The unit of t is an hour.

Epoch: 1874, *December* 8^d 11^h, *Washington Mean Time.*

VENUS.

Orbit longitude, referred to the mean equinox of date, .	$76° 58' 12''.84 + 242''.332t$
Longitude of the ascending node, ...	$75° 33' 24''.1$
Log sine of inclination, ...	8.7722486
Periodic perturbations of the latitude, ...	$+0''.11$
Log radius-vector, ...	$9.8575310 - 27.6t$
Semi-diameter at mean distance, ...	$8''.546$

THE SUN.

True longitude, referred to the mean equinox of date,	$256° 58' 41''.62 + 152''.532t$
Latitude, ...	$-0''.41$
Log radius-vector, ...	$9.9932845 - 21.3t$
Semi-diameter at mean distance, ...	$959''.788$
True obliquity of the ecliptic, ...	$23° 27' 27''.67$
Equation of the equinoxes in longitude, . . .	$-7''.42$
Sidereal time, at Washington, in arc, ...	$62° 44' 9''.6 + (15° 2' 27''.84)t$
Constant of solar parallax, ...	$8''.848$
Constant of aberration, ...	$20''.4451$
Eccentricity of the earth's meridians, ...	0.0816967
Horizontal refraction, ...	$35'$

The elements of the heliocentric position of Venus are from the new Tables of Venus,* and may be readily deduced from the first example given in pages 16–19 of the introduction.

The apparent position of the sun which results from the above elements coincides with that derived from the tables of Hansen and Olufsen, but the true longitude is $0''.19$ greater, owing to the adoption of Struve's value of the constant of aberration, $20''.445$, instead of the value $20''.255$.

The value of the sun's semi-diameter is adopted from Bessel. (See *Astronomische Nachrichten*, No. 228, and *Astronomische Untersuchungen*, Vol. II, p. 114.) This value is used in the computation of eclipses for the American Ephemeris. Hansen has also used it in his disquisition on the transit of Venus. In the British Nautical Almanac the value $961''.82$ is used, and is the same as that given for the reduction of meridian observations of the sun. Leverrier states (*Annales*, Vol. VI, p. 40) that the value, deduced from the previous transits of Venus, is $958''.424$. Hence, it is probable that predictions from the elements of the British Nautical Almanac will be found to be considerably in error from this cause.

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*Tables of Venus, prepared for the use of the American Ephemeris and Nautical Almanac, by George W. Hill, Washington, 1872.

From the data given above are obtained the following hourly ephemerides. For the sake of completeness they are expressed in terms of longitude and latitude, as well as in right ascension and declination.

VENUS.

Wash. M. T.	a = App. R. A.	δ = App. dec.	App. geocentric longitude.	App. geocentric latitude.	Log r = log distance from the earth.
1874.					
Dec. 8d 8h	255° 58′ 56″.03	−22° 38′ 9″.96	257° 4′ 53″.34	+11′ 40″.84	9.4221505
9	57 21.96	37 22.29	3 22.30	12 19.91	482
10	55 47.90	36 34.60	1 51.27	12 58.99	467
11	54 13.86	35 46.90	257 0 20.24	13 38.06	460
12	52 39.84	34 59.18	256 58 49.21	14 17.13	461
13	51 5.83	34 11.44	57 18.18	14 56.21	470
14	255 49 31.85	−22 33 23.67	256 55 47.16	+15 35.28	9.4221488

THE SUN.

Wash. M. T.	a' = App. R. A.	δ' = App. dec.	App. longitude.	App. latitude.	Log r' = log distance from the earth.
1874.					
Dec. 8d 8h	255° 42′ 16″.80	−22° 48′ 24″.39	256° 50′ 35″.86	−0″.41	9.9932909
9	45 1.47	48 39.36	53 8.39	0.41	888
10	47 46.15	48 54.28	55 40.93	0.41	867
11	50 30.84	49 9.15	256 58 13.44	0.41	845
12	53 15.54	49 23.98	257 0 45.99	0.41	824
13	56 0.25	49 38.77	3 18.52	0.41	802
14	255 58 44.98	−22 49 53.51	257 5 51.05	−0.41	9.9932781

From these quantities the position of the center of the sun, as seen from the center of Venus, is derived.

Wash. M. T.	a = R. A.	d = dec.	Log G = log distance.
1874.			
Dec. 8d 8h	255° 36′ 9″.50	−22° 52′ 9″.48	9.8575394
11	255 49 8.82	54 3.58	309
14	256 2 8.52	55 56.63	227

In the next place are obtained the following quantities, which are designated by the eclipse notation* of Chauvenet's Spherical and Practical Astronomy, which, for the most part, is identical with that of Bessel's *Analyse der Finsternisse.* It must be remembered that Venus here takes the place of the moon.

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* The plane of reference passes through the center of the earth perpendicular to the axis of the enveloping cones; a and d are the right ascension and declination of the vanishing point of the axis; μ_1, the hour-angle of that point at the first meridian; G, the distance of the sun and planet; x, y, the coordinates of the axis in the plane of reference, y being taken positive toward the north, x positive toward that point whose right ascension is $90° + a$; $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the hourly changes of x and y; f is the angle of the cone; l, the radius of the cone in the plane of reference; $i = \tan f$.

Wash. M. T.	x	$\frac{dx}{dt}$	y	$\frac{dy}{dt}$	μ_1
1874.					
Dec. 8d 8h	+37.6744	−9.74895	+25.0318	+2.59020	122° 0′ 36″.6
11	+ 8.4134	9.75838	32.7602	2.56207	166 55 0.8
14	−20.8759	−9.76782	+40.4042	+2.53393	211 49 24.6

Wash. M. T.	Exterior contacts.				Interior contacts.			
1874.	f	l	$\log l$	$\log i$	f	l	$\log l$	$\log i$
Dec. 8d 8h	22′ 24″.272	41.1254	1.614110	7.8141	22′ 0″.545	38.4845	1.585286	7.8063
11	.299	62	19	41	.570	54	296	63
14	.324	68	24	41	.595	59	301	63

CURVES REPRESENTED ON THE CHARTS.

Having now the necessary data, I proceed to explain the computations which have been made for the purpose of drawing the charts. These charts are designed to give the principal circumstances attending each of the four contacts at any point of the earth's surface where it is visible. These circumstances may be taken to be the time at which the contact occurs, and the position of the point of contact on the sun's limb. Hence, two classes of curves have been plotted on the charts—first, *curves upon which contact occurs at the same instant;* and, secondly, *curves upon which contact takes place at the same point on the sun's limb.* These curves are evidently limited, in both directions, by the curve upon which contact takes place in the horizon. The readiest method of drawing them will be to compute a sufficient number of positions conveniently distributed on these curves, and through these positions, plotted on the chart, draw the curves.

As convenient formulas for the purpose are not found in the treatises on practical astronomy, I will develop them here.

It will be amply sufficient to determine the position of these curves to within a minute of arc. Hence, as the horizontal parallax of Venus is only 33″, the effect of parallax on the right ascension and declination of the point of contact may be neglected. Then the position of this point can be found by the equations,

$$a' = a \pm \frac{s}{s' \pm s}(a' - a),$$

$$d' = \delta \pm \frac{s}{s' \pm s}(\delta' - \delta),$$

the upper sign being used for the exterior contacts, and the lower for the interior. With sufficient approximation, these equations may be written

$$a' = a \pm \frac{1}{30}(a' - a),$$

$$d' = \delta \pm \frac{1}{30}(\delta' - \delta).$$

The exterior contacts last about 21 minutes on the earth's surface, and the interior contacts about 25 minutes. The quantities a' and d' vary so slowly that they may be computed for the middle of the duration of each contact on the earth's surface, and then supposed constant for this duration. In this way the following values have been obtained:

	Wash. M. T.	a'	d'
	h m	° ′	° ′
For exterior contact at ingress........	8 40	255 57	—22 38
For interior contact at ingress........	9 10	255 57	22 37
For interior contact at egress..........	12 48	255 51	22 34
For exterior contact at egress.........	13 18	255 51	—22 34

The investigation to be made is conveniently divided into two problems.

PROBLEM I.—*To find the point of the earth's surface at which contact takes place at a given time and at a given altitude.*

Let

$\omega =$ the longitude of the required point west from the first meridian;
$\varphi =$ its latitude;
$\mu =$ the sidereal time at the first meridian;
$h =$ the given altitude;
$\theta =$ the parallactic angle at the point of contact;
$\vartheta' = \mu - a' - \omega =$ the hour-angle of the point of contact.

The general formulas of spherical trigonometry, applied to the triangle formed by the zenith, the pole, and the point of contact, give these equations:

$$\cos\varphi \sin\vartheta' = \cos h \sin\theta,$$
$$\cos\varphi \cos\vartheta' = \cos d' \sin h - \sin d' \cos h \cos\theta,$$
$$\sin\varphi = \sin d' \sin h + \cos d' \cos h \cos\theta.$$

As soon as θ is known, these three equations, together with the equation,

$$\omega = \mu - a' - \vartheta' = \mu_1' - \vartheta'$$

give the position of the required point. To obtain θ, resort must be had to the equation defining the condition of contact, viz.:

$$(l - i\zeta)^2 = (x - \xi)^2 + (y - \eta)^2,$$
$$= x^2 + y^2 - 2(x\xi + y\eta) + \rho^2 - \zeta^2.$$

In place of x and y make the usual substitutions,

$$x = m \sin M,$$
$$y = m \cos M,$$

then

$$\xi \sin M + \eta \cos M = \frac{m^2 - (l - i\zeta)^2 + \rho^2 - \zeta^2}{2m}.$$

The numerical value of each member of this equation is always less than unity, and it will be determined, to a sufficient degree of precision, with four decimals. The average value of the denominator $2m$ is about 80; hence, in the numerator it will be sufficiently accurate to put $\rho^2 = 1$, and $2li\zeta = 2mi\zeta$, and neglect the term $-i^2\zeta^2$; and if terms multiplied by i and e^2 are neglected, it is plain that $\zeta = \sin h$. Thus simplified, the equation becomes

$$\xi \sin M + \eta \cos M = \frac{m^2 - l^2 + 1}{2m} + i \sin h - \frac{1}{2m} \sin^2 h.$$

The right hand member of this equation is a known quantity, and it only remains to discover the expressions of ξ and η in terms of θ to have the equation determining θ.

The known expressions for ξ and η are

$$\xi = \rho \cos \varphi' \sin \vartheta,$$
$$\eta = \rho \cos d \sin \varphi' - \rho \sin d \cos \varphi' \cos \vartheta.$$

But if terms of the order of e^4 are neglected,

$$\rho \cos \varphi' = \frac{\cos \varphi}{\rho},$$
$$\rho \sin \varphi' = \frac{(1 - e^2) \sin \varphi}{\rho}.$$

Putting $\nu = a - a$, replacing ϑ by its value $\vartheta' + \nu$, and making $\cos \nu = 1$ since ν is a very small angle, the above expressions become

$$\rho\xi = \cos \varphi \sin \vartheta' + \sin \nu \cos \varphi \cos \vartheta',$$
$$\rho\eta = (1 - e^2) \cos d \sin \varphi - \sin d \cos \varphi \cos \vartheta' + \sin \nu \sin d \cos \varphi \sin \vartheta'.$$

In these equations substitute for $\cos \phi \sin \vartheta'$, $\cos \phi \cos \vartheta'$, and $\sin \phi$, their values in terms of θ, which have been given above; then

$$\rho\xi = \cos h \sin \theta - \sin \nu \sin d' \cos h \cos \theta + \sin \nu \cos d' \sin h,$$
$$\begin{aligned}\rho\eta = {} & \sin \nu \sin d \cos h \sin \theta + [\cos (d' - d) - e^2 \cos d' \cos d] \cos h \cos \theta \\ & + [\sin (d' - d) - e^2 \sin d' \cos d] \sin h.\end{aligned}$$

But since d' and d are very nearly equal, the last equation may be written

$$\begin{aligned}\rho\eta = {} & \sin \nu \sin d' \cos h \sin \theta + (1 - e^2 \cos^2 d') \cos h \cos \theta \\ & + [\sin (d' - d) - \tfrac{1}{2} e^2 \sin 2d'] \sin h.\end{aligned}$$

Put now

$$\sin x = \sin \nu \sin d', \qquad K' \sin x' = \sin (d' - d) - \tfrac{1}{2} e^2 \sin 2d',$$
$$K = 1 - e^2 \cos^2 d', \qquad K' \cos x' = \sin \nu \cos d'.$$

The quantities K, $\varkappa$, K', and $\varkappa'$ will be sensibly constant for the duration of each contact on the earth's surface. Then

$$\rho\xi = \cos h \sin\theta - \sin\varkappa \cos h \cos\theta + K' \cos\varkappa' \sin h,$$
$$\rho\eta = \sin\varkappa \cos h \sin\theta + K \cos h \cos\theta + K' \sin\varkappa' \sin h.$$

Since $\varkappa$ is very small and K nearly unity, there results from these equations,

$$\rho[\xi \sin M + \eta \cos M] = \sin(M+\varkappa) \cos h \sin\theta + K\cos(M+\varkappa)\cos h \cos\theta + K' \sin(M+\varkappa') \sin h.$$

In the next place make

$$L \sin(M+\lambda) = \sin(M+\varkappa),$$
$$L \cos(M+\lambda) = K \cos(M+\varkappa),$$

from which may be derived the sufficiently approximate values

$$L = 1 - e^2 \cos^2 d' \cos^2 M,$$
$$\lambda = \tfrac{1}{2}e^2 \cos^2 d' \sin 2M + \nu \sin d',$$

from which it appears that, since M does not vary much, L and λ are sensibly constant for the duration of each contact on the earth's surface. Then putting

$$\gamma = M + \lambda,$$
$$\rho[\xi \sin M + \eta \cos M] = L \cos h \cos(\theta - \gamma) + K' \sin(M+\varkappa') \sin h,$$

substituting for $\xi \sin M + \eta \cos M$, its value,

$$\frac{m^2 - l^2 + 1}{2m} + i \sin h - \frac{1}{2m} \sin^2 h,$$

and making

$$A = \frac{m^2 - l^2 + 1}{2Lm},$$

$$B = \frac{i - K' \sin(M+\varkappa')}{L},$$

$$C = -\frac{1}{2Lm},$$

and remembering that ρ and unity may be considered equal when multiplying a small term, the final equation for determining θ is

$$\cos(\theta - \gamma) = \rho \sec h [A + B \sin h + C \sin^2 h].$$

This equation possesses the advantage of having its terms separated into factors, one of which depends on the time only, and the other on the altitude only. Thus, in computing the positions of a series of points on a curve of the first class, the quantities A, B, C, and γ, since they are func-

tions of the time only, remain constant. B may be regarded as sensibly constant for the duration of each contact on the earth's surface, and C is nearly so. A, C, and γ are tabulated at intervals of a minute for the duration of each contact on the earth's surface.

The right hand member of the above equation contains the unknown factor ρ; in a first approximation this will be put equal to unity, and the value of θ thus obtained substituted in the equation,

$$\sin\varphi = \sin d' \sin h + \cos d' \cos h \cos\theta .$$

Then a sufficiently accurate value of ρ is given by the equation,

$$\rho = 1 - \tfrac{1}{2}e^2 \sin^2\varphi .$$

However, as four-place logarithms are amply sufficient for all these computations, and the means of estimating the value of ϕ to within a degree or two are usually not wanting, the repetition of the computation can be avoided. The equation gives two values for θ, corresponding to two points on the earth's surface, which satisfy the conditions of the problem, and ρ must be determined separately for each.

It remains to discover the limits between which the time and the altitude must lie, in order that the solution may be possible. It is evident that when, for a given time, h has its maximum value, the equation determining θ becomes

$$\cos(\theta - \gamma) = \pm 1 .$$

Thus the condition of contact taking place at maximum altitude is

$$\cos h = \pm\rho\,[A + B \sin h + C \sin^2 h] .$$

The ambiguous sign must be so taken that $\cos h$ may be positive. If ρ is put equal to unity or regarded as known, this equation will be of the fourth degree in $\sin h$; but since h must be in the first quadrant, it will be found, in general, to have but one root applicable to the problem. It is readily solved by successive approximations; a first value of h may be derived from $\cos h = \pm A$. According as the upper or lower sign has place, the value of θ is γ or $\gamma + 180°$. In this case of maximum altitude, the two solutions of the problem become identical.

Since each curve of contact at the same instant must have two points for $h = 0$, it follows that the time must be so taken that the numerical value of $A\rho$ may not exceed unity. Thus the equations,

$$A\rho = \pm 1 ,$$

give the times of first and last appearance of the contact on the surface of the earth.

In the special case of contact on the horizon, $h = 0$, the equation determining θ takes the simple form

$$\cos(\theta - \gamma) = A\rho,$$

and the equations determining the position of each point reduce to

$$\begin{aligned} \cos\varphi \sin\vartheta' &= \sin\theta, \\ \cos\varphi \cos\vartheta' &= -\sin d' \cos\theta, \\ \sin\varphi &= \cos d' \cos\theta, \\ \omega &= \mu_1' - \vartheta'. \end{aligned}$$

It is worthy of remark that the equation determining θ remains the same if h, instead of being exactly zero, is a small positive or negative angle; for $\sec h$ will be sensibly unity, and, B and C being small, the terms $B \sin h$ and $C \sin^2 h$ may be neglected. Hence, in taking into account the effect of refraction on the position of points, where contact takes place in the horizon, θ may still be derived from the equation,

$$\cos(\theta - \gamma) = A\rho,$$

but it will be necessary to make $h = -$ (the horizontal refraction) in the equations determining ϕ and ϑ'.

The particular case where $h = 90°$, or contact in the zenith, requires notice. Here the equation determining θ reduces to

$$A + B + C = 0.$$

This determines the time at which the phenomenon takes place; and the equations for the position of the point reduce to

$$\begin{aligned} \omega &= \mu_1', \\ \varphi &= d'. \end{aligned}$$

Problem II.—*To find the point of the earth's surface at which contact takes place at a given point on the sun's limb and at a given altitude.*

If the angle of position of the given point measured from the north point of the sun's limb toward the east is denoted by Q, the fundamental eclipse equations are

$$\begin{aligned} (l - i\zeta) \sin Q &= x - \xi, \\ (l - i\zeta) \cos Q &= y - \eta. \end{aligned}$$

In these equations $\sin h$ can be substituted for ζ, and x and y can, with sufficient approximation, be represented by the expressions,

$$x = x_0 + \frac{dx}{dt}t,$$

$$y = y_0 + \frac{dy}{dt}t,$$

if t is counted from an epoch near the middle of the duration of the contact on the earth's surface. Putting now

$$x_0 = m_0 \sin M_0, \qquad \frac{dx}{dt} = n \sin N,$$

$$y_0 = m_0 \cos M_0, \qquad \frac{dy}{dt} = n \cos N,$$

we have

$$\xi = m_0 \sin M_0 - (l - i \sin h) \sin Q + nt \sin N,$$
$$\eta = m_0 \cos M_0 - (l - i \sin h) \cos Q + nt \cos N.$$

From these equations are derived the following,

$$\xi \cos N - \eta \sin N = m_0 \sin (M_0 - N) - (l - i \sin h) \sin (Q - N),$$
$$nt = \xi \sin N + \eta \cos N - m_0 \cos (M_0 - N) + (l - i \sin h) \cos (Q - N).$$

The values of ξ and η found in the first problem must be substituted in these equations. The first member of the first of these equations is obtained simply by writing $N + 90°$ for M in the first member of the corresponding equation of the first problem. Hence, making

$$L' = 1 - e^2 \cos^2 d' \sin^2 N,$$
$$\lambda' = -\tfrac{1}{2}e^2 \cos^2 d' \sin 2N + \nu \sin d',$$
$$\gamma' = N + \lambda' + 90°,$$

these quantities are constant for the duration of each contact on the earth's surface, and there is obtained the equation

$$\rho[\xi \cos N - \eta \sin N] = L' \cos h \cos(\theta - \gamma') + K' \cos(N + \varkappa') \sin h.$$

Consequently, if

$$A' = \frac{m_0}{L'} \sin (M_0 - N) - \frac{l}{L'} \sin (Q - N),$$

$$B' = \frac{i}{L'} \sin (M_0 - N) - \frac{K'}{L'} \cos (N + \varkappa'),$$

where Q has been put equal to M_0 in the term multiplied by i, the equation determining θ in this problem becomes

$$\cos (\theta - \gamma') = \rho \sec h [A' + B' \sin h].$$

The equation giving the value of nt is only needed for the purpose of obtaining μ_1', which it is necessary to have in order to get ω from ϑ'. In this it will be sufficiently accurate to put for ξ and η their approximate values,

$$\xi = \cos h \sin \theta,$$
$$\eta = \cos h \sin \theta,$$

and neglect the term multiplied by i; then

$$nt = \cos h \cos (\theta - N) - m_0 \cos (M_0 - N) + l \cos (Q - N).$$

If μ_0 denote the value of μ_1' at the epoch from which t is counted, μ' the motion of μ_1' in a unit of time, and

$$A'' = \mu_0 - \frac{m_0 \mu'}{n} \cos (M_0 - N) + \frac{l\mu'}{n} \cos (Q - N),$$

the expression for μ_1' is

$$\mu_1' = A'' + \frac{\mu'}{n} \cos h \cos (\theta - N).$$

After θ and μ_1' have been determined from the equations just given, the position of the point on the earth's surface is found by means of the same equations as in the first problem. Thus it appears that the solutions of the two problems are quite similar, the only differences being that the term corresponding to $C \sin^2 h$, in the factor of the equation which determines θ, is wanting, and that a separate computation must be made for μ_1'; and the remarks to be made regarding the solution of the equation determining θ, and the limits between which Q and h must be assumed, in order that solution may be possible, are quite similar to those made in the first problem. While B' and γ' are constant for the duration of each contact on the earth's surface, A' and A'' involve the variable Q, and may be tabulated with Q as the argument within its limiting values. The equation determining θ gives two values for this quantity, corresponding to the two points on the earth's surface, which satisfy the conditions of the problem; and ρ must be determined separately for each.

The condition of contact taking place at a given point on the sun's limb, and at the maximum altitude, is

$$\cos h = \pm \rho [A' + B' \sin h],$$

and the equations

$$A'\rho = \pm 1,$$

give the limiting values of Q. In finding the points on the curves of the second class, which are common to the curve of contact on the horizon, θ is derived from the equation

$$\cos(\theta-\gamma')=A'\rho\,,$$

but $h=-$ (the horizontal refraction) in the equations which determine ϕ and ϑ'. In computing the value of μ_1' for each of the two solutions of the problem, it will be noticed that, with sufficient approximation, the second term has the same numerical value but opposite signs in the two solutions; and, in the case of maximum altitude for a given value of Q, the equation becomes simply

$$\mu_1'=A''.$$

In this case also, it will be advantageous to compute four auxiliary quantities from the equations,

$$p\cos\varepsilon=\cos d', \qquad p'\sin\varepsilon'=\sin d',$$
$$p\sin\varepsilon=\sin d'\cos\theta, \quad p'\cos\varepsilon'=\cos d'\cos\theta,$$

by means of which the equations determining ϕ and ϑ' take the form,

$$\cos\varphi\sin\vartheta'=\cos h\sin\theta\,,$$
$$\cos\varphi\cos\vartheta'=p\sin(h-\varepsilon)\,,$$
$$\sin\varphi\ =p'\cos(h-\varepsilon')\,.$$

As θ is constant in this case, $p, p', \varepsilon, \varepsilon'$, are so likewise, provided that after the point of maximum altitude has passed the zenith, h be supposed to increase from 90° to 180°, or, in other words, that $180°-h$ be used instead of h.

VALUES OF THE QUANTITIES EMPLOYED.

Denoting the four contacts in their order by the symbols I, II, III, and IV, the values of the various quantities employed in the foregoing discussion are:

	I	II	III	IV
Epoch from which t is counted,	$8^h\,40^m$	$9^h\,10^m$	$12^h\,48^m$	$13^h\,18^m$
ν, ...	$+18'$	$+16'$	$-6'$	$-8'$
$\log K'$, ...	7.9187	7.9158	7.9359	7.9417
$\varkappa'$, ...	54° 20′	58° 33′	100° 46′	104° 14′
$\log L$, ...	9.9989	9.9986	9.9977	9.9977
λ, ...	$+3'$	$+4'$	$-3'$	$-1'$
$\log B$, ...	n7.1880	7.2228	n7.3475	n7.3411
N, ...	284° 50′ 30″.5	284° 48′ 49″.5	284° 36′ 36″.5	284° 34′ 55″.6

	I	II	III	IV
L', ...	9.9977	9.9977	9.9977	9.9977
λ', ...	$-2'$	$-1'$	$+7'$	$+8'$
$\frac{m_0}{L'}\sin(M_0-N)$, ...	+34.0289	+34.0187	+34.0188	+34.0290
$\log\left[-\frac{l}{L'}\right]$, ...	1.616412	n1.587591	n1.587600	1.616423
γ', ...	14° 49′	14° 48′	14° 44′	14° 43′
$\log B'$, ...	7.3788	7.3517	7.3349	n7.3605
$\mu_0-\frac{m_0\mu'}{n}\cos(M_0-N)$, . . .	166° 23′	166° 26′	166° 43′	166° 45′
$\log\left[\frac{l\mu'}{n}\text{in minutes of arc}\right]$, . .	3.5657	3.5369	3.5368	3.5656
$\log\left[\frac{\mu'}{n}\text{in minutes of arc}\right]$, .	1.9516	1.9516	1.9515	1.9515
$\log p$, ...	9.99758	9.9978	9.9979	9.9979
ε, ...	21° 7′	21° 56′	−21° 54′	−21° 54′
$\log p'$, ...	9.9876	9.9876	9.9877	9.9877
ε', ...	−156° 40′	−156° 41′	−23° 15′	−23° 15′

The quantities which vary with the time and with Q are given in the following tables.

I.—*For Exterior Contact at Ingress.*

Wash. M. T.	A	log C	γ	μ'_1	Wash. M. T.	A	log C	γ	μ'_1
h m			° ′	° ′	h m			° ′	° ′
8 29	+1.0339	n 8.0752	51 29	128 56	8 40	−0.0313	n 8.0864	49 25	131 41
30	0.9360	.0762	51 18	129 11	41	0.1268	.0874	49 13	131 56
31	0.8383	.0772	51 7	129 26	42	0.2220	.0884	49 1	132 11
32	0.7408	.0782	50 56	129 41	43	0.3170	.0894	48 50	132 27
33	0.6435	.0793	50 44	129 56	44	0.4117	.0904	48 38	132 42
34	0.5464	.0803	50 33	130 11	45	0.5061	.0914	48 26	132 57
35	0.4495	.0813	50 22	130 26	46	0.6003	.0924	48 14	133 12
36	0.3529	.0823	50 11	130 41	47	0.6942	.0934	48 2	133 27
37	0.2565	.0833	49 59	130 56	48	0.7878	.0943	47 50	133 42
38	0.1603	.0844	49 48	131 11	49	0.8811	.0953	47 38	133 57
39	+0.0644	.0854	49 36	131 26	50	0.9742	.0963	47 26	134 12
8 40	−0.0313	n 8.0864	49 25	131 41	8 51	−1.0670	n 8.0973	47 14	134 27

Q	A'	A''	Q	A'	A''	Q	A'	A''
46° 50′	−1.0360	133° 54′	48° 30′	−0.3841	132° 26′	50° 10′	+0.2969	130° 58′
47 0	0.9721	133 45	40	0.3173	132 17	20	0.3666	130 49
10	0.9079	133 36	50	0.2502	132 8	30	0.4366	130 40
20	0.8435	133 27	49 0	0.1828	131 59	40	0.5069	130 31
30	0.7788	133 19	10	0.1152	131 51	50	0.5774	130 23
40	0.7138	133 10	20	−0.0472	131 42	51 0	0.6482	130 14
50	0.6484	133 1	30	+0.0211	131 33	10	0.7193	130 5
48 0	0.5827	132 52	40	0.0897	131 24	20	0.7907	129 56
10	0.5168	132 43	50	0.1585	131 15	30	0.8624	129 47
20	0.4506	132 35	50 0	0.2276	131 7	40	0.9343	129 39
48 30	−0.3841	132 26	50 10	+0.2969	130 58	51 50	+1.0065	129 30

II.—*For Interior Contact at Ingress.*

Wash. M. T.	A	log C	γ	μ'_1	Wash. M. T.	A	log C	γ	μ'_1
h m			° '	° '	h m			° '	° '
8 57	+1.0501	n 8.1034	46 1	135 57	9 10	—0.0238	n 8.1155	43 13	139 13
58	0.9651	.1044	45 48	136 12	11	0.1037	.1164	43 0	139 28
59	0.8807	.1053	45 36	136 27	12	0.1832	.1173	42 47	139 43
9 0	0.7967	.1063	45 23	136 42	13	0.2623	.1181	42 33	139 58
1	0.7130	.1072	45 10	136 57	14	0.3410	.1190	42 20	140 13
2	0.6296	.1081	44 57	137 12	15	0.4193	.1199	42 7	140 28
3	0.5466	.1091	44 45	137 27	16	0.4972	.1208	41 53	140 43
4	0.4640	.1100	44 32	137 42	17	0.5746	.1217	41 39	140 58
5	0.3817	.1109	44 19	137 57	18	0.6515	.1225	41 26	141 13
6	0.2998	.1118	44 6	138 12	19	0.7280	.1234	41 12	141 28
7	0.2183	.1127	43 53	138 28	20	0.8040	.1243	40 58	141 43
8	0.1372	.1137	43 39	138 43	21	0.8795	.1251	40 44	141 58
9	+0.0565	.1146	43 26	138 58	22	0.9546	.1260	40 30	142 13
9 10	—0.0238	n 8.1155	43 13	139 13	9 23	—1.0292	n 8.1268	40 17	142 28

Q	A'	A''	Q	A'	A''	Q	A'	A''
39° 50'	—1.0401	142° 10'	42° 0'	—0.3965	140° 13'	44° 10'	+0.2964	138° 19'
40 0	0.9924	142 1	10	0.3449	140 4	20	0.3517	138 10
10	0.9444	141 52	20	0.2930	139 55	30	0.4073	138 2
20	0.8961	141 43	30	0.2409	139 47	40	0.4632	137 53
30	0.8474	141 34	40	0.1885	139 38	50	0.5194	137 44
40	0.7985	141 25	50	0.1358	139 29	45 0	0.5758	137 35
50	0.7493	141 16	43 0	0.0827	139 20	10	0.6325	137 26
41 0	0.6997	141 7	10	—0.0294	139 11	20	0.6895	137 18
10	0.6499	140 58	20	+0.0242	139 3	30	0.7468	137 9
20	0.5998	140 49	30	0.0781	138 54	40	0.8044	137 1
30	0.5494	140 40	40	0.1322	138 45	50	0.8623	136 52
40	0.4987	140 31	50	0.1866	138 37	46 0	0.9204	136 44
50	0.4477	140 22	44 0	0.2414	138 28	10	0.9788	136 35
42 0	—0.3965	140 13	44 10	+0.2964	138 19	46 20	+1.0375	136 27

III.—*For Interior Contact at Egress.*

Wash. M. T.	A	log C	γ	μ'_1	Wash. M. T.	A	log C	γ	μ'_1
h m			° '	° '	h m			° '	° '
12 35	—1.0188	n 8.1276	—10 53	190 41	12 48	—0.0100	n 8.1162	—13 50	193 57
36	0.9440	.1268	11 7	190 56	49	+0.0706	.1153	14 3	194 12
37	0.8687	.1259	11 21	191 11	50	0.1515	.1144	14 16	194 27
38	0.7929	.1251	11 34	191 26	51	0.2329	.1135	14 29	194 42
39	0.7166	.1242	11 48	191 41	52	0.3147	.1126	14 42	194 57
40	0.6399	.1233	12 2	191 56	53	0.3968	.1116	14 55	195 12
41	0.5627	.1224	12 16	192 11	54	0.4793	.1107	15 8	195 27
42	0.4850	.1216	12 29	192 26	55	0.5622	.1098	15 21	195 42
43	0.4069	.1207	12 43	192 41	56	0.6454	.1089	15 34	195 57
44	0.3284	.1198	12 56	192 56	57	0.7290	.1079	15 46	196 12
45	0.2494	.1189	13 9	193 11	58	0.8129	.1070	15 59	196 27
46	0.1700	.1180	13 23	193 27	12 59	0.8972	.1060	16 11	196 42
47	0.0902	.1171	13 36	193 42	13 0	0.9819	.1051	16 24	196 57
12 48	—0.0100	n 8.1162	—13 50	193 57	13 1	+1.0669	n 8.1041	—16 36	197 12

Q	A′	A″	Q	A′	A″	Q	A′	A″
° ′		° ′	° ′		° ′	° ′		° ′
—10 30	—1.0148	191 4	—12 40	—0.3690	193 0	—14 50	+0.3258	194 55
40	0.9669	191 13	50	0.3173	193 9	15 0	0.3813	195 4
50	0.9187	191 22	13 0	0.2653	193 18	10	0.4371	195 13
11 0	0.8702	191 31	10	0.2131	193 27	20	0.4931	195 21
10	0.8215	191 40	20	0.1605	193 36	30	0.5494	195 30
20	0.7724	191 49	30	0.1076	193 44	40	0.6059	195 39
30	0.7230	191 58	40	0.0545	193 53	50	0.6628	195 47
40	0.6733	192 7	50	—0.0010	194 2	16 0	0.7200	195 56
50	0.6233	192 16	14 0	+0.0528	194 11	10	0.7775	196 5
12 0	0.5731	192 25	10	0.1069	194 20	20	0.8353	196 13
10	0.5226	192 34	20	0.1612	194 29	30	0.8933	196 22
20	0.4717	192 43	30	0.2158	194 37	40	0.9515	196 30
30	0.4205	192 51	40	0.2706	194 46	—16 50	+1.0101	196 39
—12 40	—0.3690	193 0	—14 50	+0.3258	194 55			

IV.—*For Exterior Contact at Egress.*

Wash. M. T.	A	log C	γ	μ′₁	Wash. M. T.	A	log C	γ	μ′₁
h m			° ′	° ′	h m			° ′	° ′
13 7	—1.0536	n 8.0983	—17 48	198 43	13 18	—0.0144	n 8.0874	—19 59	201 29
8	0.9605	.0974	18 1	198 58	19	+0.0817	.0864	20 10	201 44
9	0.8671	.0964	18 13	199 13	20	0.1780	.0854	20 22	201 59
10	0.7735	.0954	18 25	199 28	21	0.2745	.0844	20 33	202 14
11	0.6795	.0944	18 37	199 44	22	0.3713	.0834	20 45	202 29
12	0.5853	.0934	18 49	199 59	23	0.4683	.0823	20 56	202 44
13	0.4908	.0924	19 0	200 14	24	0.5655	.0813	21 8	202 59
14	0.3960	.0914	19 12	200 29	25	0.6629	.0803	21 19	203 14
15	0.3010	.0904	19 24	200 44	26	0.7605	.0793	21 30	203 29
16	0.2057	.0894	19 36	200 59	27	0.8583	.0783	21 41	203 44
17	0.1102	.0884	19 47	201 14	28	0.9563	.0772	21 53	203 59
13 18	—0.0144	n 8.0874	—19 59	201 29	13 29	+1.0545	n 8.0762	—22 4	204 14

Q	A′	A″	Q	A′	A″	Q	A′	A″
° ′		° ′	° ′		° ′	° ′		°
—17 30	—1.0021	199 18	—19 10	—0.3487	200 48	—20 50	+0.3341	202 16
40	0.9380	199 27	20	0.2817	200 57	21 0	0.4039	202 25
50	0.8737	199 36	30	0.2144	201 6	10	0.4739	202 34
18 0	0.8091	199 45	40	0.1468	201 15	20	0.5443	202 42
10	0.7443	199 54	50	0.0790	201 24	30	0.6150	202 51
20	0.6791	200 3	20 0	—0.0109	201 33	40	0.6859	203 0
30	0.6136	200 12	10	+0.0575	201 41	50	0.7572	203 8
40	0.5478	200 21	20	0.1262	201 50	22 0	0.8288	203 17
50	0.4817	200 30	30	0.1952	201 59	10	0.9006	203 26
19 0	0.4153	200 39	40	0.2645	202 8	20	0.9727	203 34
—19 10	—0.3487	200 48	—20 50	+0.3341	202 16	—22 30	+1.0451	203 43

BEGINNING, ETC., OF EACH CONTACT.

From the foregoing data are readily derived the times, and position of the places, at which the following phenomena occur.

		Wash. M. T.	Longitude.	Latitude.
		h m	° ′	° ′
Contact I	begins on the earth	8 29.335	55 27	+35 24
	occurs in the zenith	8 39.530	131 34	—22 38
	ends on the earth	8 50.292	244 25	—38 24
Contact II	begins on the earth	8 57.572	65 53	+40 15
	occurs in the zenith	9 9.520	139 6	—22 37
	ends on the earth	9 22.630	257 24	—44 22
Contact III	begins on the earth	12 35.216	36 40	—64 33
	occurs in the zenith	12 48.314	194 2	—22 34
	ends on the earth	13 0.244	235 18	+62 48
Contact IV	begins on the earth	13 7.548	58 15	—61 0
	occurs in the zenith	13 18.300	201 33	—22 34
	ends on the earth	13 28.471	251 17	+59 20

APPROXIMATION OF THE CURVES TO CIRCLES.

The curves to be drawn on the charts approximate so closely to circles of the sphere that it has been deemed sufficient to compute the positions of three points on each curve, namely, the two at which contact occurs on the horizon, and the one for which the altitude is a maximum, and then regard the curve as a circle of the sphere passing through these points; and, as the stereographic projection has been chosen for the delineation of the charts, the projected curves will also be circles.

But it will be of interest to determine beforehand how great an error can be produced by this assumption. And first, in the case of the time-curves, let σ be the radius of the circle of the sphere passing through the three points, and adopt the subscripts (0), (1), (2), (3), for the quantities which refer respectively to the pole of this circle, the points of contact on the horizon, and the point of maximum altitude. Then σ and the position of the pole of this circle are determined by the equations,

$$\begin{aligned}
\sin\varphi_1 \sin\varphi_0 + \cos\varphi_1 \cos\varphi_0 \cos(\vartheta'_1 - \vartheta'_0) &= \cos\sigma, \\
\sin\varphi_2 \sin\varphi_0 + \cos\varphi_2 \cos\varphi_0 \cos(\vartheta'_2 - \vartheta'_0) &= \cos\sigma, \\
\sin\varphi_3 \sin\varphi_0 + \cos\varphi_3 \cos\varphi_0 \cos(\vartheta'_3 - \vartheta'_0) &= \cos\sigma,
\end{aligned}$$

or, if for the moment we write, in general,

$$\begin{aligned}
x &= \cos\varphi \sin\vartheta', \\
y &= \cos\varphi \cos\vartheta', \\
z &= \sin\varphi,
\end{aligned}$$

by the equations,

$$\begin{aligned}
x_1 x_0 + y_1 y_0 + z_1 z_0 &= \cos\sigma, \\
x_2 x_0 + y_2 y_0 + z_2 z_0 &= \cos\sigma, \\
x_3 x_0 + y_3 y_0 + z_3 z_0 &= \cos\sigma.
\end{aligned}$$

It will be sufficient to assume here that the circle which passes through the point of maximum altitude and the two points for which $h = -$ (the horizontal refraction) will also pass through the two points for which $h = 0$. Consequently, we shall suppose that $h_1 = 0$ and $h_2 = 0$. But, from the foregoing investigation,

$$\begin{array}{lll} x_1 = \sin\theta_1, & x_2 = \sin\theta_2, & x_3 = \cos h_3 \sin\theta_3, \\ y_1 = -\sin d' \cos\theta_1, & y_2 = -\sin d' \cos\theta_2, & y_3 = \cos d' \sin h_3 - \sin d' \cos h_3 \cos\theta_3, \\ z_1 = \cos d' \cos\theta_1, & z_2 = \cos d' \cos\theta_2, & z_3 = \sin d' \sin h_3 + \cos d' \cos h_3 \cos\theta_3, \end{array}$$

and if two unknowns, υ and τ, are taken so that

$$x_0 = \sin\upsilon,$$
$$y_0 = -\sin(d' - \tau)\cos\upsilon,$$
$$z_0 = \cos(d' - \tau)\cos\upsilon,$$

the equations determining σ, υ, and τ are

$$\sin\theta_1 \sin\upsilon + \cos\theta_1 \cos\upsilon\cos\tau = \cos\sigma,$$
$$\sin\theta_2 \sin\upsilon + \cos\theta_2 \cos\upsilon\cos\tau = \cos\sigma,$$
$$\cos h_3 \sin\theta_3 \sin\upsilon + \cos h_3 \cos\theta_3 \cos\upsilon \cos\tau + \sin h_3 \cos\upsilon \sin\tau = \cos\sigma,$$

from which are derived

$$\sec\sigma \sin\upsilon = \frac{\sin\frac{1}{2}(\theta_2 + \theta_1)}{\cos\frac{1}{2}(\theta_2 - \theta_1)},$$

$$\sec\sigma \cos\upsilon \cos\tau = \frac{\cos\frac{1}{2}(\theta_2 + \theta_1)}{\cos\frac{1}{2}(\theta_2 - \theta_1)},$$

$$\sec\sigma \cos\upsilon \sin\tau = \frac{1}{\sin h_3}\left[1 - \cos h_3 \frac{\cos\frac{1}{2}(\theta_2 + \theta_1 - 2\theta_3)}{\cos\frac{1}{2}(\theta_2 - \theta_1)}\right].$$

But since θ_1 and θ_2 are given by the equations,

$$\cos(\theta_1 - \gamma) = A\rho_1 = A\left(1 - \frac{e^2}{2}\cos^2 d' \cos^2\theta_1\right),$$

$$\cos(\theta_2 - \gamma) = A\rho_2 = A\left(1 - \frac{e^2}{2}\cos^2 d' \cos^2\theta_2\right),$$

where $\theta_2 - \gamma$ is nearly $360° - \theta_1 + \gamma$, we shall have

$$\cos\left[\frac{\theta_2 + \theta_1}{2} - \gamma\right] \cos\frac{\theta_2 - \theta_1}{2} = \pm A\left[1 - \frac{e^2}{2}\cos^2 d' \,(A^2 \cos 2\gamma + \sin^2\gamma)\right]$$

$$\sin\left[\frac{\theta_2 + \theta_1}{2} - \gamma\right] = \mp \tfrac{1}{2}A^2 e^2 \cos^2 d' \sin 2\gamma.$$

As for the ambiguous signs, they are determined by the following conditions: Let it be agreed that the position of the pole, for which σ is less

than 90°, is to be found. And as the equations ought not to be changed when the subscripts (1) and (2) are interchanged, let

$$\beta = \frac{\theta_2 \sim \theta_1}{2},$$

be so taken that β is in the first quadrant, and let

$$\gamma_0 = \frac{\theta_2 + \theta_1}{2},$$

be taken in that quadrant which makes $\frac{\theta_2 + \theta_1}{2} - \theta_3$ a small positive, or negative, angle; then

$$\gamma_0 = \gamma - \tfrac{1}{2} A^2 e^2 \cos^2 d' \sin 2\gamma$$

when A is positive, and this expression augmented by 180°, when A is negative; and

$$\cos \beta = \pm A \left[1 - \frac{e^2}{2} \cos^2 d' (A^2 \cos 2\gamma + \sin^2 \gamma) \right],$$

the ambiguous sign to be so taken that $\cos \beta$ may be positive. The quantity $\cos \left(\frac{\theta_2 + \theta_1}{2} - \theta_3 \right)$ differs from unity by a quantity of the order of e^4, which may be neglected. Moreover, h_3 and β are nearly equal. Thus, the equations determining σ, υ, and τ take the simpler forms,

$$\sec \sigma \sin \upsilon = \frac{\sin \gamma_0}{\cos \beta},$$

$$\sec \sigma \cos \upsilon \cos \tau = \frac{\cos \gamma_0}{\cos \beta},$$

$$\sec \sigma \cos \upsilon \sin \tau = \frac{h_3 - \beta}{\cos \beta}.$$

Since $h_3 - \beta$ is small, its square may be neglected, and the equations give

$$\sigma = \beta,$$
$$\upsilon = \gamma_0,$$
$$\tau = \frac{h_3 - \beta}{\cos \gamma_0},$$

whence τ is a small positive or negative angle. The position of the pole of the circle is then given by the equations,

$$\begin{aligned} \cos \varphi_0 \cos \vartheta_0' &= - \cos \gamma_0 \sin (d' - \tau), \\ \cos \varphi_0 \sin \vartheta_0 &= \sin \gamma_0, \\ \sin \varphi_0 &= \cos \gamma_0 \cos (d' - \tau), \\ \omega_0 &= \mu_1' - \vartheta_0'. \end{aligned}$$

If the distance of any point on the time-curve from this pole be denoted by σ', then $\sigma' - \sigma$ may be taken as a sufficiently exact measure of the error committed by our method of drawing the curve.

But

$$\begin{aligned} \cos\sigma' &= xx_0 + yy_0 + zz_0 , \\ x &= \cos h \sin \theta , \\ y &= \cos d' \sin h - \sin d' \cos h \cos \theta , \\ z &= \sin d' \sin h + \cos d' \cos h \cos \theta , \end{aligned}$$

whence

$$\cos\sigma' = \cos h \sin\theta \sin\gamma_0 + \cos h \cos\theta \cos\gamma_0 \cos\tau + \sin h \cos\gamma_0 \sin\tau ,$$

or, as $\cos\tau$ may be put equal to unity,

$$\cos\sigma' = (h_3 - \beta) \sin h + \cos h \cos(\theta - \gamma_0) .$$

The quantity $\sigma' - \sigma$ is composed of two parts independent of each other; the first depending on the curvature of the cone enveloping the sun and Venus, and proportional to the quantity we have denoted by C; the second due to the non-sphericity of the earth and proportional to e^2. These parts can then be determined separately.

First, from the equations,

$$\begin{aligned} \cos h_3 &= \pm (A + B \sin h_3 + C \sin^2 h_3) , \\ \cos\beta &= \pm A , \end{aligned}$$

is obtained, with sufficient exactness,

$$h_3 - \beta = \mp (B + C \sin\beta) .$$

But

$$\begin{aligned} \cos h \cos(\theta - \gamma_0) &= \pm (A + B \sin h + C \sin^2 h) , \\ \cos\sigma &= \pm A , \end{aligned}$$

thus

$$\cos\sigma' - \cos\sigma = \pm C \sin h (\sin h - \sin\beta) .$$

Secondly, from the equations,

$$\cos h_3 = \pm A\rho_3 = \pm A \left[1 - \frac{e^2}{2}(\sin d' \sin\beta \pm \cos d' \cos\beta \cos\gamma)^2\right],$$

$$\cos\beta = \pm A \left[1 - \tfrac{1}{2} e^2 \cos^2 d' \, (A^2 \cos 2\gamma + \sin^2\gamma)\right],$$

we find that the part of $h_3 - \beta$ proportional to e^2 is

$$h_3 - \beta = \tfrac{1}{2} e^2 \cos\beta \left[\sin^2 d' \sin\beta \pm \sin 2d' \cos\beta \cos\gamma - \cos^2 d' \sin\beta \sin^2\gamma\right].$$

Also

$$\cos h \cos (\theta - \gamma_0) = \pm A\rho \mp \tfrac{1}{2} e^2 \cos^2 d' \sin 2\gamma \cos^2 \beta \cos h \sin (\theta - \gamma)\,,$$

$$= \pm A \left[1 - \frac{e^2}{2} (\sin d' \sin h + \cos d' \cos h \cos \theta)^2 \right],$$

$$\mp \frac{e^2}{2} \cos^2 d' \sin 2\gamma \cos^2 \beta \sqrt{(\sin^2 \beta - \sin^2 h)}\,,$$

$$\cos h \cos \theta = A \cos \gamma - \sqrt{(\sin^2 \beta - \sin^2 h)} \sin \gamma\,,$$

where the sign of $\sin (\theta - \gamma)$ must be attributed to the radical $\sqrt{(\sin^2 \beta - \sin^2 h)}$.

After some reductions it will be found that

$$\cos \sigma' - \cos \sigma = \frac{e^2}{2} (\sin^2 d' - \cos^2 d' \sin^2 \gamma) \cos \beta \sin h (\sin \beta - \sin h)$$
$$+ \frac{e^2}{2} \sin 2d' \sin \gamma \cos \beta \sin h \sqrt{(\sin^2 \beta - \sin^2 h)}\,.$$

Uniting to this the term proportional to C, we have the complete value

$$\cos \sigma' - \cos \sigma = \left[\frac{e^2}{2} \sin^2 d' - \cos^2 d' \sin^2 \gamma) \cos \beta \mp C \right] (\sin \beta - \sin h) \sin h$$
$$+ \frac{e^2}{2} \sin 2d' \sin \gamma \cos \beta \sin h \sqrt{(\sin^2 \beta - \sin^2 h)}\,.$$

It will be seen that this expression vanishes when $h = 0$ and when $h = \beta$, as it should. Differentiating it with respect to the variable $\sin h$, in order to obtain its maximum value, we arrive at an equation of the fourth degree in $\sin h$. Hence we are obliged to content ourselves with a superior limit to the maximum value, which, however, for practical purposes, may be regarded as identical with it. The first term of the expression has its maximum value when $\sin h = \frac{1}{2} \sin \beta$, and the second when $\sin h = \frac{1}{\sqrt{2}}$ $\sin \beta$. Substituting these values in their respective terms, we obtain

$$\sigma' - \sigma = \frac{e^2}{16} (\cos^2 d' \sin^2 \gamma \pm 2 \sin 2d' \sin \gamma - \sin^2 d') \sin 2\beta \pm \frac{C}{4} \sin \beta\,,$$

where the ambiguous signs, in both cases, must be so taken that the largest numerical value of the expression will be obtained. Replacing e^2 and C by their values, and taking for the factor which involves d' and γ the greatest value it can have, it results that $\sigma' - \sigma$ cannot exceed

$$11' \sin \beta + 2' \sin 2\beta\,,$$

and the maximum value of this with regard to the variable β is less than 12.′ Having regard to the scale on which the charts have been constructed, this

quantity may be considered as within the unavoidable errors produced by imperfection of drawing.

It is worthy of remark that, in our method of drawing the curves, the error is only a fourth part of that which results from neglecting altogether the curvature of the cones enveloping the sun and the planet, as has generally been done in treatises on practical astronomy.

The investigation of the error in the case of the second class of curves differs somewhat from that of the first class, on account of μ_1' not being constant for all points on the curve. The equations determining σ and the position of the pole are

$$\begin{aligned}
\sin\varphi_1 \sin\varphi_0 + \cos\varphi_1 \cos\varphi_0 \cos(\omega_1 - \omega_0) &= \cos\sigma\,,\\
\sin\varphi_2 \sin\varphi_0 + \cos\varphi_2 \cos\varphi_0 \cos(\omega_2 - \omega_0) &= \cos\sigma\,,\\
\sin\varphi_3 \sin\varphi_0 + \cos\varphi_3 \cos\varphi_0 \cos(\omega_3 - \omega_0) &= \cos\sigma\,,
\end{aligned}$$

where

$$\begin{aligned}
\omega_1 &= A'' - \frac{\mu'}{n}\sin(\theta_1 - \gamma') - \vartheta_1'\,,\\
\omega_2 &= A'' + \frac{\mu'}{n}\sin(\theta_1 - \gamma') - \vartheta_2'\,,\\
\omega_3 &= A'' - \vartheta_3'\,.
\end{aligned}$$

If we put

$$\begin{aligned}
g &= \frac{\mu'}{n}\sin(\theta_1 - \gamma') = \pm\frac{\mu'}{n}\sqrt{(1 - A'^2)}\,,\\
\omega_0 &= A'' - \vartheta_0'\,,
\end{aligned}$$

g is a small angle, whose square may be neglected, and the equations, using the notation given in the case of the first class of curves, take the shape

$$\begin{aligned}
(x_1 + gy_1)\,x_0 + (y_1 - gx_1)\,y_0 + z_1 z_0 &= \cos\sigma\,,\\
(x_2 - gy_2)\,x_0 + (y_2 + gx_2)\,y_0 + z_2 z_0 &= \cos\sigma\,,\\
x_3 x_0 + y_3 y_0 + z_3 z_0 &= \cos\sigma\,.
\end{aligned}$$

Put

$$\begin{aligned}
\theta_1' &= \theta_1 - g\sin d'\,,\\
\theta_2' &= \theta_2 + g\sin d'\,,
\end{aligned}$$

and, as τ is here also a small angle, making $\cos\tau = 1$, the equations, using the same notation as before, become

$$\begin{aligned}
\sin\theta_1' \sin\upsilon + \cos\theta_1' \cos\upsilon &= \cos\sigma\,,\\
\sin\theta_2' \sin\upsilon + \cos\theta_2' \cos\upsilon &= \cos\sigma\,,\\
\cos h_3 \sin\theta_3 \sin\upsilon + \cos h_3 \cos\theta_3 \cos\upsilon + \sin h_3 \cos\upsilon \sin\tau &= \cos\sigma\,.
\end{aligned}$$

These are entirely similar to the analogous equations for the first class of curves. Hence, the operations here being identical with those of the former case, it will be necessary to note only the final results. If

$$\gamma_0 = \gamma' - \tfrac{1}{2}A'e^2\cos^2 d'\sin 2\gamma'$$

when A' is positive, and this expression augmented by 180° when A' is negative, and

$$\cos\beta = \pm A'\left[1-\frac{e^2}{2}\cos^2 d'\,(A''^2\cos 2\gamma'+\sin^2\gamma')\right] \pm \frac{\mu'}{n}(1-A'^2)\sin d',$$

the upper or lower signs being taken so as to render $\cos\beta$ positive, then

$$\sigma = \beta,$$
$$\tau = \frac{h_3-\beta}{\cos\gamma_0},$$

and the position of the pole of the circle is given by the equations,

$$\begin{aligned}
\cos\varphi_0\sin\vartheta_0' &= \sin\gamma_0,\\
\cos\varphi_0\cos\vartheta_0' &= -\cos\gamma_0\sin(d'-\tau),\\
\sin\varphi_0 &= \cos\gamma_0\cos(d'-\tau),\\
\omega_0 &= A''-\vartheta_0'.
\end{aligned}$$

To determine the error of representing this class of curves by circles of the sphere,

$$\cos\sigma' = \sin\varphi\sin\varphi_0+\cos\varphi\cos\varphi_0\cos(\omega-\omega_0),$$
$$\omega = A''-\frac{\mu'}{n}\cos h\sin(\theta-\gamma')-\vartheta',$$

whence

$$\begin{aligned}
\cos\sigma' &= xx_0+yy_0+zz_0+\frac{\mu'}{n}\cos h\sin(\theta-\gamma')(yx_0-xy_0),\\
&= (h_3-\beta)\sin h+\cos h\cos(\theta-\gamma_0)\\
&\quad +\frac{\mu'}{n}\sqrt{(\sin^2\beta-\sin^2 h)}\,[\cos d'\sin\gamma_0\sin h+\sin d'\cos h\sin(\theta-\gamma_0)],\\
&= (h_3-\beta)\sin h+\cos h\cos(\theta-\gamma_0)\pm\frac{\mu'}{n}\sin d'\,(\sin^2\beta-\sin^2 h)\\
&\quad \pm\frac{\mu'}{n}\cos d'\sin\gamma'\sin h\sqrt{(\sin^2\beta-\sin^2 h)},
\end{aligned}$$

where the upper or lower sign is taken according as A' is positive or negative, and the sign of $\sin(\theta-\gamma')$ is assigned to the radical

$$\sqrt{(\sin^2\beta-\sin^2 h)}.$$

The part of $\cos\sigma'-\cos\sigma$ which involves the factor $\frac{\mu'}{n}$ will be found to be

$$\pm\frac{\mu'}{n}\sin d'\sin h\,(\sin\beta-\sin h),$$
$$\pm\frac{\mu'}{n}\cos d'\sin\gamma'\sin h\sqrt{(\sin^2\beta-\sin^2 h)}.$$

The part proportional to e^2 is obtained from the analogous expression in the case of the first class of curves, simply by changing γ into γ', and thus is

$$\frac{e^2}{2}(\sin^2 d'-\cos^2 d'\sin^2\gamma')\cos\beta\sin h\,(\sin\beta-\sin h)$$
$$+\frac{e^2}{2}\sin 2d'\sin\gamma'\cos\beta\sin h\sqrt{(\sin^2\beta-\sin^2 h)}.$$

Combining these two parts, we have

$$\cos\sigma' - \cos\sigma = \left[\frac{e^2}{2}(\sin^2 d' - \cos^2 d' \sin^2 \gamma')\cos\beta \pm \frac{\mu'}{n}\sin d'\right](\sin\beta - \sin h)\sin h,$$

$$+\left[\frac{e^2}{2}\sin 2d' \sin\gamma' \cos\beta \pm \frac{\mu'}{n}\cos d' \sin\gamma'\right]\sin h\sqrt{(\sin^2\beta - \sin^2 h)}.$$

Deriving a superior limit to the maximum value of $\sigma' - \sigma$ by the same method as in the former case, it is found to be, with regard to the variable h,

$$\sigma' - \sigma = -\frac{e^2}{16}(\sin^2 d' \pm 2\sin 2d' \sin\gamma' - \cos^2 d' \sin^2\gamma')\sin 2\beta$$
$$\pm\frac{\mu'}{4n}(\sin d' \pm 2\cos d' \sin\gamma')\sin\beta,$$

where the ambiguous signs must be taken so as to make the numerical value of the expression the largest. On substituting the numerical values of d' and γ', it will be seen that the term proportional to e^2 has no appreciable effect in augmenting the maximum value of $\sigma' - \sigma$, which is found to be 18′.

POSITIONS OF POINTS OF THE CURVES.

The positions of the points needed for drawing the curves are given below; for the two points on the horizon $h = -35'$; and for the point of maximum altitude, the value of this quantity is given in the last column.

I.—*Exterior Contact at Ingress.*

FIRST CLASS OF CURVES.

	Contact on the horizon.				Contact at maximum altitude.		
Wash.M.T.	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
h m	° ′	° ′	° ′	° ′	° ′	° ′	° ′
8 30	71 42	+53 8	45 47	+16 52	76 32	+23 37	21 2
31	88 46	61 56	41 9	+ 5 37	87 4	15 41	33 32
32	107 51	66 27	37 56	— 2 40	94 22	9 32	42 45
33	128 12	67 56	35 14	9 37	100 23	+ 4 14	50 34
34	146 54	67 3	32 45	15 48	105 42	— 0 34	57 32
36	173 35	61 7	27 48	26 48	115 18	9 14	70 5
38	189 18	52 40	22 19	36 37	124 27	17 3	81 33
40	199 29	43 19	15 25	45 40	133 50	24 18	87 25
42	207 4	33 18	5 45	54 4	144 9	31 8	76 24
44	213 24	22 36	350 40	61 25	156 15	37 22	64 58
46	219 20	10 42	325 43	66 17	171 29	42 50	52 28
47	222 25	+ 3 59	308 46	66 43	181 0	45 1	45 27
48	225 50	— 3 36	290 27	64 58	192 25	46 32	37 31
49	230 1	12 46	272 38	60 14	206 51	46 52	27 49
8 50	237 1	—26 41	254 41	—49 38	228 10	—43 57	12 47

SECOND CLASS OF CURVES.

Angle of position of point of contact.	Contact on the horizon.				Contact at maximum altitude.		
Wash. M.T.	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
47° 0′	259° 46′	−53° 33′	311° 52′	−66° 46′	252° 16′	−73° 36′	14° 10′
47 20	244 42	38 21	351 59	60 57	182 55	73 32	32 37
47 40	237 41	27 52	7 26	52 52	158 46	65 0	44 28
48 0	232 58	18 55	16 1	45 1	148 31	56 25	54 19
48 20	229 7	10 47	21 46	37 29	142 35	48 19	63 8
48 40	225 39	− 3 10	26 5	30 11	138 27	40 31	71 23
49 0	222 20	+ 4 11	29 37	23 2	135 13	32 55	79 20
49 20	219 1	11 24	32 44	15 54	132 30	25 23	87 10
49 40	215 29	18 37	35 37	8 41	130 3	17 48	85 0
50 0	211 36	25 54	38 29	− 1 19	127 46	10 1	76 59
50 20	207 2	33 24	41 28	+ 6 24	125 28	− 1 55	68 38
50 40	201 16	41 14	44 49	14 38	123 3	+ 6 47	59 41
51 0	193 11	49 34	48 57	23 43	120 19	16 26	49 45
51 20	179 34	58 32	54 46	34 20	116 45	27 51	37 56
51 40	147 8	+67 4	66 17	+48 37	110 22	+43 47	21 16

II.—*Interior Contact at Ingress.*

FIRST CLASS OF CURVES.

	Contact on the horizon.				Contact at maximum altitude.		
Wash.M.T.	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
h m 8 58	78° 40′	+53° 6′	57° 25′	+26° 25′	82° 32′	+31° 1′	15° 42′
59	96 45	62 17	52 8	14 50	93 49	22 24	28 46
9 0	115 41	66 34	48 57	6 54	100 51	16 5	37 46
1	135 13	67 57	46 28	+ 0 22	106 20	10 45	45 8
2	152 56	67 12	44 20	− 5 21	111 3	+ 5 57	51 40
4	178 21	61 57	40 27	15 23	119 11	− 2 36	63 7
6	193 36	54 37	36 40	24 16	126 31	10 17	73 21
8	203 36	46 37	32 36	32 24	133 37	17 26	82 56
10	210 52	38 17	27 48	40 5	140 52	24 13	87 48
12	216 41	29 45	21 43	47 25	148 41	30 42	78 38
14	221 45	20 53	13 17	54 24	157 33	36 56	69 18
16	226 32	11 26	0 41	60 45	168 15	42 51	59 30
18	231 24	+ 1 4	340 39	65 33	182 3	48 11	48 47
19	234 8	− 4 52	326 47	66 42	190 51	50 27	42 47
20	237 5	11 10	310 30	66 23	201 28	52 11	36 5
21	240 45	18 42	293 3	63 53	214 42	52 57	28 10
9 22	246 12	−28 48	275 6	−57 43	232 36	−51 42	17 20

SECOND CLASS OF CURVES.

Angle of position of point of contact.	Contact on the horizon.				Contact at maximum altitude.		
	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
40° 0′	275° 5′	−57° 43′	305° 33′	−65° 56′	274° 58′	−69° 48′	8° 15′
40 20	256 41	43 33	349 9	64 4	212 8	75 58	26 37
40 40	249 44	34 27	6 45	58 10	179 26	70 41	37 7
41 0	245 5	26 50	16 36	51 59	165 11	64 1	45 38
41 20	241 26	20 1	23 3	46 1	157 24	57 30	53 7
41 40	238 17	13 41	27 45	40 9	152 16	51 10	60 2
42 0	235 26	7 40	31 25	34 29	148 30	45 5	66 33
42 20	232 45	− 1 50	34 27	28 53	145 29	39 5	72 52
42 40	230 7	+ 3 51	37 7	23 19	142 58	33 12	79 1
43 0	227 28	9 30	39 31	17 45	140 44	27 19	85 8
43 20	224 43	15 8	41 47	12 8	138 42	21 24	88 48
43 40	221 48	20 50	43 57	6 25	136 47	15 23	82 32
44 0	218 35	26 36	46 9	− 0 33	134 55	9 13	76 10
44 20	214 55	32 34	48 27	+ 5 34	133 2	− 2 46	69 32
44 40	210 31	38 45	51 0	12 1	131 6	+ 4 2	62 32
45 0	204 57	45 13	53 55	18 57	128 59	11 22	54 59
45 20	197 14	52 6	57 33	26 37	126 33	19 32	46 34
45 40	184 50	59 24	62 36	35 29	123 26	29 8	36 38
46 0	159 13	+66 27	71 37	+46 59	118 23	+41 49	23 22

III.—*Interior Contact at Egress.*

FIRST CLASS OF CURVES.

	Contact on the horizon.				Contact at maximum altitude.		
Wash. M. T.	Log.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
h m 12 36	67° 18′	−52° 11′	349° 56′	−65° 24′	86° 11′	−79° 10′	19° 26′
37	76 50	43 42	330 38	60 36	137 51	77 40	29 31
38	82 28	36 50	320 13	55 34	159 44	72 18	37 11
39	86 37	30 45	313 26	50 46	169 51	66 37	43 49
40	89 57	25 11	308 35	46 10	175 36	61 19	49 39
42	95 23	14 59	301 48	37 24	182 34	51 18	60 16
44	100 0	− 5 30	297 4	29 7	187 2	41 55	70 2
46	104 18	+ 3 33	293 21	21 4	190 32	32 53	79 22
48	108 34	12 28	290 8	13 3	193 34	23 59	88 33
50	113 7	21 22	287 8	− 4 54	196 27	14 57	82 9
52	118 21	30 28	284 5	+ 3 36	199 21	− 5 35	72 30
54	124 57	39 57	280 44	12 41	202 29	+ 4 26	62 9
56	134 33	50 2	276 33	22 51	206 9	15 38	50 31
57	141 44	55 19	273 50	28 38	208 24	22 0	43 52
58	152 26	60 44	270 13	35 12	211 12	29 18	36 13
59	170 41	65 49	264 43	43 10	215 9	38 10	26 50
13 0	209 26	+67 32	251 50	+55 3	223 42	+52 6	11 45

SECOND CLASS OF CURVES.

Angle of position of point of contact.	Contact on the horizon.				Contact at maximum altitude.		
	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
—10° 40′	316° 1′	—52° 46′	12° 28′	—66° 50′	306° 6′	—74° 19′	15° 20′
11 0	304 19	40 59	45 8	62 36	250 1	74 59	29 43
11 20	298 50	32 24	60 29	56 30	224 56	68 57	39 30
11 40	295 7	25 2	69 45	50 22	213 42	62 15	47 41
12 0	292 15	18 23	76 14	44 24	207 27	55 45	55 0
12 20	289 48	12 8	81 10	38 36	203 23	49 30	61 47
12 40	287 35	6 10	85 13	32 56	200 27	43 25	68 15
13 0	285 31	— 0 23	88 44	27 20	198 13	37 28	74 31
13 20	283 29	+ 5 18	91 53	21 47	196 23	31 35	80 39
13 40	281 24	10 57	94 48	16 13	194 48	25 42	86 45
14 0	279 13	16 36	97 37	10 34	193 24	19 45	87 6
14 20	276 50	22 18	100 22	— 4 49	192 6	13 42	80 51
14 40	274 5	28 8	103 10	+ 1 7	190 49	7 28	74 26
15 0	270 53	34 8	106 6	7 19	189 32	— 0 55	67 42
15 20	266 49	40 24	109 17	13 52	188 8	+ 6 0	60 35
15 40	261 25	46 58	112 56	20 58	186 34	13 31	52 50
16 0	253 24	53 59	117 26	28 54	184 34	21 59	44 6
16 20	239 27	61 22	123 41	38 15	181 43	32 9	33 34
—16 40	206 43	+67 44	135 51	+51 6	175 56	+46 36	18 22

IV.—*Exterior Contact at Egress.*

FIRST CLASS OF CURVES.

	Contact on the horizon.				Contact at maximum altitude.		
Wash. M. T.	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
h m 13 8	79° 5′	—49° 12′	22° 19′	—66° 49′	92° 22′	—71° 58′	16° 27′
9	89 39	37 40	350 56	64 5	136 44	72 10	29 44
10	95 17	29 12	335 3	59 2	159 56	67 18	38 58
11	99 23	21 58	325 34	53 46	172 21	61 35	46 42
12	102 43	15 28	319 2	48 27	180 2	55 52	53 35
14	108 21	— 3 35	310 30	38 19	189 34	44 54	65 56
16	113 25	+ 7 22	304 48	28 34	195 52	34 25	77 20
18	118 31	17 53	300 20	18 54	200 52	24 7	88 21
20	124 11	28 19	296 27	— 9 7	205 21	13 42	80 34
22	131 13	38 54	292 37	+ 1 12	209 48	— 2 47	68 59
24	141 26	49 54	288 21	12 30	214 41	+ 9 9	56 16
25	149 7	55 33	285 46	18 49	217 33	15 48	49 9
26	160 34	61 11	282 33	25 55	220 59	23 17	41 5
27	179 43	66 10	278 5	34 18	225 35	32 8	31 27
13 28	215 33	+67 37	269 28	+46 2	233 48	+44 25	17 40

SECOND CLASS OF CURVES.

Angle of position of point of contact.	Contact on the horizon.				Contact at maximum altitude.		
	Long.	Lat.	Long.	Lat.	Long.	Lat.	Max. alt.
−17° 40′	319° 8′	−48° 33′	33° 31′	−66° 10′	294° 52′	−76° 11′	20° 36′
18 0	308 37	35 23	63 48	58 49	239 29	71 16	36 16
18 20	303 17	25 27	77 16	50 43	222 6	62 39	47 14
18 40	299 26	16 44	85 26	42 56	214 8	54 12	56 43
19 0	296 21	8 50	91 17	35 27	209 26	46 8	65 23
19 20	293 33	− 1 18	95 55	28 12	206 13	38 24	73 32
19 40	290 50	+ 6 2	99 54	21 3	203 47	30 50	81 26
20 0	288 5	13 14	103 30	13 54	201 46	23 17	89 15
20 20	285 5	20 27	106 56	− 6 40	199 58	15 40	82 53
20 40	281 39	27 48	110 24	+ 0 48	198 17	− 7 49	74 48
21 0	277 26	35 22	114 1	8 37	196 33	+ 0 25	66 19
21 20	271 49	43 20	118 6	17 2	194 40	9 19	57 10
21 40	263 21	51 51	123 7	26 26	192 23	19 19	46 51
22 0	247 33	60 59	130 20	37 42	189 2	31 29	34 15
−22 20	202 9	+68 1	147 35	+54 33	180 40	+50 42	14 0

Explanation of the Charts and their Use.

These charts are the development by the stereographic projection of a sphere four inches in diameter. The center of each chart is in latitude 22° 54′ south, and the border is at the distance of 100° from this zenith. The charts are so placed in longitude that the illuminated hemisphere, at the time of contact, may occupy the central portion of the chart. The longitudes, which are noted along the equator, are counted westerly from the meridian of Washington; and, to avoid the inconvenience of repeating the same figures as eastern longitudes, which might give rise to provoking errors in the use of these charts, the numeration has been carried beyond 180° up to 360°. The latitudes are noted along the middle meridian. The smaller islands of the Pacific and Indian Oceans have been indicated on the charts only in those regions which are favorably situated for observations for determining the parallax; and no geographical names have been placed on the charts, except the names of a few principal cities and towns and islands in the same regions.

As to the curves delineated on the charts, one will notice, first, the continuous line in black limiting all the other broken lines, and on which is inscribed "Contact on the horizon." At all points on this line the phenomenon of contact, mentioned in the title of the chart, will take place in the horizon; that is, the point of contact common to the limbs of Venus and the sun will be in the horizon. As stated above, the horizontal refraction

has been allowed for in determining the position of this curve. This curve, then, limits the part of the earth's surface in which the phenomenon is visible. At all points on the curve to the right of the middle meridian of the chart, the sun will be setting at the time of the phenomenon, and at all points to the left it will be rising.

At any station within this curve, the altitude of Venus and the sun will be approximately equal to the arc of a great circle drawn from the station to meet the curve at right angles. On each chart will be found a "Scale for altitudes." If, having plotted any station on the chart, we measure, with a pair of dividers, the shortest distance to the curve of "Contact on the horizon," and then apply this distance to the scale, we shall get the altitude of the point of contact, with an error of not more than $\frac{1}{4}^\circ$, or, in extreme cases, $\frac{1}{2}^\circ$, an approximation sufficient for the purpose of estimating the extinction of light. The scale is limited to 30°, as, beyond this point, the extinction of light is not of importance.

The azimuth of the point of contact, differing but little from that of the sun's center, may be estimated from the chart by drawing a line from the station to the central point of the chart, which lies on the middle meridian in latitude 22° 54′ south, and measuring the angle which this line makes with the meridian of the station. To avoid drawing this meridian, we may take points in the same latitude on the two nearest meridians, find the azimuth at each, and then, by interpolation, the azimuth at the station.

One will notice, next, the dotted lines *in blue*. At all points, on each of these, the phenomenon of contact takes place at the same instant, the corresponding Washington mean time of which is noted at the right hand extremity of the line. It will be noticed that there is an interval of one minute between the times corresponding to the first five and last five curves on the chart, but that elsewhere the interval is two minutes. In addition to these curves, the positions, situated on the curve of "Contact on the horizon," where the phenomenon of contact makes its first and last appearance on the earth's surface, are indicated, as are also the times of their occurrence.

In the region crossed by the curves passing through the central part of the chart, where the intervals between the successive curves are nearly equal, one will have no difficulty in interpolating between them. Having plotted the station on the chart by its given longitude, counted west from Washington, and its latitude, conceive a line to be drawn through it, perpendicular to each of the adjacent curves, and, having ascertained the proportion of the parts into which the station divides this line, find the time

which divides the interval between the times belonging to the adjacent curves in the same ratio. This will be the Washington mean time of the contact at the station. Subtracting from this the longitude west from Washington, converted into time, one will get the local mean time of contact It may be well to notice here that in Eastern Europe, Asia, Africa, Australia, and New Zealand, where the time has been arrived at in going eastward from Europe, the transit occurs, in civil time, on December 9, but that in the Sandwich and other islands of the Pacific where the time has been arrived at in going westward from America and Europe, the transit occurs on December 8.

In the regions near the points of first and last appearance of contact, interpolation between the curves is more difficult, owing to the irregularity of the intervals. A satisfactory result, however, can be obtained from using the principle that the interval between the times, corresponding to two time-curves, is nearly proportional to the difference of the cosines of the maximum altitudes, at which contact occurs on these curves. This is best illustrated by an example. Let it be required to find the time of interior contact at egress, at Khiva, in the region east of the Caspian Sea. On referring to chart No. 3, the time is seen to lie between $12^h\,59^m$ and $13^h\,0^m$. From the tables given above, we find that the maximum altitudes at which contact occurs on these time-curves are, respectively, $26°\,50'$ and $11°\,45'$. Interpolating between these, as in the preceding case, we shall find that the maximum altitude of the time-curve which passes through Khiva is about $23°\,8'$. Then the required time is given by the following expression :

$$12^h\,59^m + \frac{\cos 23°\;\;8' - \cos 26°\,50'}{\cos 11°\,45' - \cos 26°\,50'} \times 1^m,$$

and this is $12^h\,59^m\,19^s$. When the station at which it is desired to find the time of contact lies within the first or last time-curve drawn on the chart, the point of first or last appearance of contact on the earth's surface, with its associated time, takes the place in the interpolation of one of the time-curves. The error of the time of contact, derived in this way, ought not to exceed 5^s. However, for stations in the central portions of the chart, it may sometimes be a little more. In this error we must be understood as including only errors of drawing, plotting, and measuring upon the chart, and not errors in the elements, on which the computations for the charts have been based, the effect of which may be very much larger.

One will notice, lastly, the dotted lines *in red.* At all points, on each of these, contact occurs at the same point on the sun's limb. The angle of

position of this point, counted from the north point of the limb toward the east, for the two charts which belong to the ingress, but toward the west for the two which belong to the egress, is noted at the left hand extremity of each curve. The interval between the angles, corresponding to two adjacent curves, is uniformly 20′. The angle of position of the point of contact, for any station, can be found in precisely the same manner as the time from the time-curves. The greatest and least angles of position of the point of contact which occur on the curve of "Contact on the horizon" are not noted on the charts. As they may be needed in interpolation, I give them here.

	Minimum value.	Maximum value.
Chart No. 1	46° 55′.6	51° 49′.1
Chart No. 2	39 58.4	46 13.6
Chart No. 3	10 33.1	16 48.3
Chart No. 4	17 30.3	22 23.8

TABLES AND FORMULAS FOR COMPUTING TIMES OF CONTACT.

As more accurate values of the times of contact may be desired than can be derived from the charts, tables of data, entirely similar to the data for solar eclipses given in the American Ephemeris, are here appended:

I.—*Exterior Contact at Ingress.*

Wash. M. T.	*A*	*B*	*C*	*μ*
h m				
8 29	+32.9620	+67.4082	—14.8430	129° 14′.7
30	32.7995	67.4513	14.7999	129 29.7
31	32.6370	67.4944	14.7568	129 44.6
32	32.4745	67.5375	14.7137	129 59.6
33	32.3119	67.5806	14.6706	130 14.6
34	32.1494	67.6237	14.6275	130 29.5
35	31.9869	67.6667	14.5845	130 44.5
36	31.8243	67.7098	14.5414	130 59.5
37	31.6618	67.7529	14.4983	131 14.5
38	31.4993	67.7960	14.4552	131 29.4
39	31.3368	67.8390	14.4122	131 44.4
40	31.1743	67.8821	14.3691	131 59.4
41	31.0118	67.9251	14.3261	132 14.3
42	30.8493	67.9682	14.2830	132 29.3
43	30.6867	68.0113	14.2399	132 44.3
44	30.5242	68.0543	14.1969	132 59.2
45	30.3617	68.0974	14.1538	133 14.2
46	30.1992	68.1404	14.1108	133 29.2
47	30.0367	68.1835	14.0677	133 44.1
48	29.8741	68.2265	14.0247	133 59.1
49	29.7116	68.2696	13.9816	134 14.1
50	29.5491	68.3126	13.9386	134 29.0
8 51	+29.3866	+68.3557	—13.8955	134 44.0

II.—*Interior Contact at Ingress.*

Wash. M. T. h m	A	B	C	μ
8 57	+28.4115	+65.9731	−10.9967	136° 13′.8
58	28.2490	66.0162	10.9536	136 28.8
59	28.0865	66.0592	10.9106	136 43.8
9 0	27.9239	66.1022	10.8676	136 58.7
1	27.7614	66.1452	10.8246	137 13.7
2	27.5988	66.1882	10.7816	137 28.7
3	27.4363	66.2312	10.7386	137 43.6
4	27.2737	66.2742	10.6956	137 58.6
5	27.1112	66.3172	10.6526	138 13.6
6	26.9487	66.3602	10.6096	138 28.6
7	26.7861	66.4032	10.5666	138 43.5
8	26.6236	66.4462	10.5236	138 58.5
9	26.4610	66.4892	10.4806	139 13.5
10	26.2985	66.5322	10.4376	139 28.4
11	26.1360	66.5752	10.3946	139 43.4
12	25.9734	66.6182	10.3516	139 58.4
13	25.8108	66.6611	10.3087	140 13.3
14	25.6482	66.7041	10.2657	140 28.3
15	25.4857	66.7471	10.2227	140 43.3
16	25.3232	66.7901	10.1797	140 58.2
17	25.1606	66.8330	10.1368	141 13.2
18	24.9981	66.8760	10.0938	141 28.2
19	24.8355	66.9189	10.0509	141 43.1
20	24.6730	66.9619	10.0079	141 58.1
21	24.5104	67.0049	9.9649	142 13.1
22	24.3479	67.0478	9.9220	142 28.1
9 23	+24.1853	+67.0908	− 9.8790	142 43.0

III.—*Interior Contact at Egress.*

Wash. M. T. h m	A	B	C	μ
12 35	− 7.0414	+75.2908	−1.6806	190° 37′.0
36	7.2041	75.3332	1.6382	190 52.0
37	7.3668	75.3757	1.5957	191 7.0
38	7.5296	75.4181	1.5533	191 22.0
39	7.6923	75.4606	1.5108	191 36.9
40	7.8550	75.5030	1.4684	191 51.9
41	8.0177	75.5454	1.4260	192 6.9
42	8.1804	75.5879	1.3835	192 21.8
43	8.3432	75.6303	1.3411	192 36.8
44	8.5059	75.6728	1.2986	192 51.8
45	8.6686	75.7152	1.2562	193 6.7
46	8.8313	75.7576	1.2138	193 21.7
47	8.9941	75.8000	1.1714	193 36.7
48	9.1568	75.8425	1.1289	193 51.6
49	9.3196	75.8849	1.0865	194 6.6
50	9.4823	75.9273	1.0441	194 21.6
51	9.6450	75.9697	1.0017	194 36.6
52	9.8078	76.0121	0.9593	194 51.5
53	9.9705	76.0546	0.9168	195 6.5
54	10.1333	76.0970	0.8744	195 21.5
55	10.2960	76.1394	0.8320	195 36.4
56	10.4587	76.1818	0.7896	195 51.4
57	10.6215	76.2242	0.7472	196 6.4
58	10.7842	76.2665	0.7049	196 21.3
59	10.9470	76.3089	0.6625	196 36.3
13 0	11.1097	76.3513	0.6201	196 51.3
13 1	−11.2724	+76.3937	−0.5777	197 6.2

IV.—*Exterior Contact at Egress.*

Wash. M. T. h m	A	B	C	μ
13 7	—12.2489	+79.2889	—2.9645	198° 36′.1
8	12.4117	79.3313	2.9221	198 51.0
9	12.5744	79.3736	2.8798	199 6.0
10	12.7372	79.4160	2.8374	199 21.0
11	12.9000	79.4584	2.7950	199 35.9
12	13.0627	79.5007	2.7527	199 50.9
13	13.2255	79.5431	2.7103	200 5.9
14	13.3882	79.5854	2.6680	200 20.8
15	13.5510	79.6278	2.6256	200 35.8
16	13.7138	79.6701	2.5833	200 50.8
17	13.8765	79.7125	2.5409	201 5.7
18	14.0393	79.7548	2.4986	201 20.7
19	14.2020	79.7972	2.4563	201 35.7
20	14.3648	79.8395	2.4139	201 50.6
21	14.5276	79.8818	2.3716	202 5.6
22	14.6903	79.9242	2.3292	202 20.6
23	14.8531	79.9665	2.2869	202 35.6
24	15.0158	80.0089	2.2445	202 50.5
25	15.1786	80.0512	2.2022	203 5.5
26	15.3414	80.0935	2.1599	203 20.5
27	15.5041	80.1358	2.1176	203 35.4
28	15.6669	80.1782	2.0752	203 50.4
13 29	—15.8296	+80.2205	—2.0329	204 5.4

The other quantities needed for the computation may be taken to be constant for each contact, and have the following values:

	log E	log F	log G	log H	A'	B' and C'
I	9.96323	9.96562	*n* 9.59632	*n* 9.58290	—27.087	+7.177
II	9.96323	9.96558	*n* 9.59629	*n* 9.58312	—27.091	+7.164
III	9.96311	9.96546	*n* 9.59697	*n* 9.58382	—27.122	+7.070
IV	9.96307	9.96546	*n* 9.59718	*n* 9.58379	—27.127	+7.057

A', B', and C' are respectively the variations of A, B, and C in one second, and are expressed in units of the fourth decimal place.

If the values of these quantities be taken for a time T_0, assumed near the time of contact at any place, an exact value of the time may be computed by the following formulas:

$\varphi =$ the latitude of the place, positive when north,

$\omega =$ its longitude from Washington, positive when west,

$$\log e = 8.9122, \quad \log(1-e^2) = 9.99709, \quad \sin\chi = e\sin\varphi,$$

$$h = \sec\chi\cos\varphi, \quad k = (1-e^2)\sec\chi\sin\varphi,$$

$$a = A - h\sin(\mu-\omega),$$

$$b = B - Ek + Gh\cos(\mu-\omega),$$

$$c = -C + Fk - Hh\cos(\mu-\omega),$$

$m = \sqrt{bc}$, (usually with the same sign as a).

If $m = a$, the time T_0 is correctly chosen. If m differs from a, a correction of the assumed time may be obtained in seconds, by the formulas,

$$\log \mu' = 9.8617,$$
$$\tan \tfrac{1}{2} Q = \frac{c}{m} = \frac{m}{b},$$
$$a' = A' - \mu' h \cos(\mu - \omega),$$
$$b' = B' - \mu' G h \sin(\mu - \omega),$$
$$t = \frac{10000\,(m - a)}{a' + b' \cot Q},$$

and the actual Washington time of contact will be

$$T_0 + t,$$

and the local mean time of the phenomenon will be

$$T_0 + t - \omega.$$

Q must be taken of the same sign with a, and is a sufficiently near approximation to the angular distance of the point of contact, reckoned from the north point of the sun's limb toward the east.

To find V, the angular distance of the point of contact from the *vertex* of the sun's limb, positive toward the *left*, we have the formulas,

$$p \sin P = \sin \varphi,$$
$$p \cos P = \cos \varphi \cos(\mu - \omega),$$
$$c \sin C = \cos P \tan(\mu - \omega),$$
$$c \cos C = \sin(P - \delta'),$$
$$V = Q - C,$$

in which δ' is the sun's declination.

The following is an example of the computation of the time of interior contact, at ingress, at Honolulu, Sandwich Islands. The longitude and latitude of the place are derived from the *Connaissance des Temps* for 1868:

	$\varphi = +21^\circ 18' 12''$		$\omega = 80^\circ 51' 45''$
(1)	$\log e = 8.91220$		
(2)	$\log \sin \varphi = 9.56027$	(1) + (2)	$\log \sin \chi = 8.47247$
(3)	$\log (1 - e^2) = 9.99709$		
(4)	$\log \sec \chi = 0.00019$	(2) + (3) + (4)	$\log k = 9.55755$
(5)	$\log \cos \varphi = 9.96926$	(4) + (5)	$\log h = 9.96945$

From chart No. 2 the Washington mean time of contact is found to be nearly $8^h\ 58^m\ 24^s$, which will be taken as the value of T_0.

19

Computation of t, the correction of T_0.

$\mu = 136° 32'.8$

$\mu - \omega = 55° 41'.0$

(1) $\log \sin (\mu - \omega) = 9.91695$

(2) $\log h = 9.96945$

(3) $\log \cos (\mu - \omega) = 9.75110$

(4) = (1) + (2). $\log h \sin (\mu - \omega) = 9.88640$

(5) $\log \mu' = 9.8617$

(6) $\log G = 9.59629n$

(7) = (2) + (3) $\log h \cos (\mu - \omega) = 9.72055$

(8) $\log H = 9.58312n$

(6) + (7) $\log Gh \cos (\mu - \omega) = 9.31684n$

(7) + (8) $\log Hh \cos (\mu - \omega) = 9.30367n$

(5) + (7) $\log \mu' h \cos (\mu - \omega) = 9.5822$

(4) + (5) + (6) $\log \mu' Gh \sin (\mu - \omega) = 9.3444n$

(9) $\log E = 9.96323$

(10) $\log k = 9.55755$

(11) $\log F = 9.96558$

(9) + (10) $\log Ek = 9.52078$

(10) + (11) $\log Fk = 9.52313$

(12) $A = + 28.1840$

(13) $- h \sin (\mu - \omega) = - 0.7698$

(14) $B = + 66.0334$

(15) $- Ek = - 0.3317$

(16) $Gh \cos (\mu - \omega) = - 0.2074$

(17) $- C = + 10.9364$

(18) $Fk = + 0.3335$

(19) $- Hh \cos (\mu - \omega) = + 0.2012$

(12) + (13) $a = + 27.4142$

(14) + (15) + (16) $b = + 65.4943$

(17) + (18) + (19) $c = + 11.4711$

$m = + 27.4097$

$m - a = - 0.0045$

(20) $\log b = 1.8162035$

(21) $\log c = 1.0596051$

(22) = ½ [(20) + (21)] $\log m = 1.4379043$

(22) − (20) $\log \tan \frac{1}{2} Q = 9.6217108$

Angle from north point, $Q = 45°25'10''$

(29) $\log \cot Q = 9.9936$

(30) $\log b' = 0.8686$

(29) + (30) $\log b' \cot Q = 0.8622$

(31) $\log (m - a) + 4 = 1.6532n$

(32) $\log (a' + b' \cot Q) = 1.3050n$

(31) − (32) $\log t = 0.3482$

(23) $A' = - 27.08$

(24) $- \mu' h \cos (\mu - \omega) = - 0.38$

(25) $B' = + 7.17$

(26) $- \mu' Gh \sin (\mu - \omega) = + 0.22$

(25) + (26) $b' = + 7.39$

(27) = (23) + (24) $a' = - 27.46$

(28) $b' \cot Q = + 7.28$

(27) + (28) $a' + b' \cot Q = - 20.18$

		h m s
Assumed time, ...	$T_0 =$	8 58 24.0
Correction of the assumed time, ...	$t =$	+ 2.2
Washington time of the contact, ...		8 58 26.2
Honolulu time of the contact, ...		3 34 59.2

We have also $C = 55° 1'.2$, and the angle from the *vertex*, $V = - 9° 36'.0$.

The corrections which should be applied to the times of the four contacts for determinate changes in the elements, exclusive of the effect of a change in the constant of solar parallax, are given by the following formulas. In these

$\delta\odot$ = the correction of the sun's longitude,
δL = the correction of the orbit longitude of Venus,
$\delta\Omega$ = the correction of the longitude of the node of Venus,
δB = the correction of the sun's latitnde,
δs = the correction of the semi-diameter of Venus at the mean distance,
$\delta s'$ = the correction of the semi-diameter of the sun at the mean distance.

All these quantities being expressed in seconds of arc, the corrections of the times of the four contacts, in their order, are

$$\begin{aligned}
\delta T_1 &= +\overset{s}{48.4}\,(\delta\odot - \delta L) + \overset{s}{3.00}\,(\delta L - \delta\Omega + 16.9\delta B) - \overset{s}{97.4}\,\delta s - \overset{s}{26.1}\,\delta s', \\
\delta T_2 &= +50.9\,(\delta\odot - \delta L) + 3.94\,(\delta L - \delta\Omega + 16.9\delta B) + 116.3\,\delta s - 31.2\,\delta s', \\
\delta T_3 &= +30.2\,(\delta\odot - \delta L) - 4.68\,(\delta L - \delta\Omega + 16.9\delta B) - 116.3\,\delta s + 31.2\,\delta s', \\
\delta T_4 &= +30.1\,(\delta\odot - \delta L) - 3.75\,(\delta L - \delta\Omega + 16.9\delta B) + 97.4\,\delta s + 26.1\,\delta s'.
\end{aligned}$$

These expressions have been computed for the center of the earth, but they may be taken as approximately exact for any point on the surface.

An approximate value of the co-efficient of the correction of the constant of solar parallax, for any place, may be found by subtracting from the ascertained Washington mean time of contact at the place, the Washington mean time of the same contact occurring in the zenith, given on page 128. Thus in the example for Honolulu, given above, one finds that

$$\begin{aligned}
\delta T_2 &= (8^h\,58^m\,26^s.2 - 9^h\,9^m.520)\frac{\delta\pi_0}{\pi_0}, \\
&= -665^s.0\,\frac{\delta\pi_0}{\pi_0},
\end{aligned}$$

where π_0 denotes the constant of solar parallax. It must be understood, however, that this method gives quite rude approximations.

POSITION OF THE PLANET ON THE SUN'S DISC.

All that precedes relates to the contacts; but it may be desired to find the position of the planet, when on the sun's disc, relative to the center of this body. For this purpose the following tables of data are appended.

Wash. M. T.	x	Change of x in 1 minute.	y	Change of y in 1 minute.	μ	d
h m						
8 30	+32.7995	—0.16251	+26.3257	+0.04309	129°29′.7	—22°52′.5
40	31.1744	16252	26.7565	4306	131 59.4	52.6
50	29.5492	16252	27.1870	4304	134 29.0	52.7
9 0	27.9239	16253	27.6173	4301	136 58.7	52.8
10	26.2985	16254	28.0473	4298	139 28.4	52.9
20	24.6730	16255	28.4770	4296	141 58.1	53.0
30	23.0474	16256	28.9065	4293	144 27.8	53.1
40	21.4217	16257	29.3357	4291	146 57.5	53.2
50	19.7960	16258	29.7647	4288	149 27.1	53.3
10 0	18.1702	16259	30.1934	4286	151 56.8	53.4
10	16.5443	16260	30.6219	4283	154 26.5	53.5
20	14.9183	16260	31.0501	4281	156 56.2	53.7
30	13.2922	16261	31.4780	4278	159 25.9	53.8
40	11.6660	16262	31.9057	4275	161 55.6	53.9
50	10.0397	16263	32.3331	4273	164 25.3	54.0
11 0	8.4134	16264	32.7602	4270	166 55.0	54.1
10	6.7870	16265	33.1871	4267	169 24.7	54.2
20	5.1605	16266	33.6137	4265	171 54.3	54.3
30	3.5338	16267	34.0401	4262	174 24.0	54.4
40	1.9071	16268	34.4662	4259	176 53.7	54.5
50	+ 0.2803	16268	34.8920	4257	179 23.4	54.6
12 0	— 1.3465	16269	35.3176	4254	181 53.1	54.7
10	2.9734	16270	35.7429	4252	184 22.8	54.8
20	4.6004	16271	36.1680	4249	186 52.5	54.9
30	6.2276	16272	36.5928	4247	189 22.2	55.0
40	7.8549	16273	37.0173	4244	191 51.9	55.1
50	9.4822	16274	37.4416	4242	194 21.6	55.2
13 0	11.1096	16275	37.8656	4239	196 51.3	55.3
10	12.7371	16276	38.2894	4236	199 21.0	55.4
20	14.3647	16276	38.7129	4234	201 50.6	55.5
13 30	—15.9924	—0.16277	+39.1361	+0.04231	204 20.3	—22 55.6

The distance D in seconds of arc of the center of Venus from the center of the sun, and the angle of position Q of this distance, counted from the north point toward the east, are obtained by the formulas,

$$\vartheta = \mu - \omega\,,$$
$$\varDelta \sin Q = x - \rho \cos \varphi' \sin \vartheta\,,$$
$$\varDelta \cos Q = y - \rho \sin \varphi' \cos d + \rho \cos \varphi' \sin d \cos \vartheta\,,$$
$$\log D = 1.388945 + \log \varDelta\,.$$

At the time of minimum distance of centers we have the equation,

$$(x' - \xi') \sin Q + (y' - \eta') \cos Q = 0\,.$$

If $\xi = 0$ and $\eta = 0$, the solution of this equation gives the circumstances of this phenomenon as it would be seen from the center of the earth.

With the foregoing data the Washington mean time is found to be $10^{\rm h}\ 58^{\rm m}\ 55^{\rm s}$, and

$$Q = 14°\ 42'\ 43''.3\,.$$

Since the time of minimum distance for any point on the surface of the earth cannot differ more than 6 or 7 minutes from the time of the same phenomenon for the center of the earth, we may assume that x', y', and d are constant in this problem, and have the same values as in the case of the center of the earth. Introducing, then, a small auxiliary angle E, determined by the equation

$$\tan E = \frac{[8.0703]\rho \cos \varphi' \sin(\vartheta - 34^\circ\, 1')}{1 + [8.4010]\rho \cos \varphi' \cos(\vartheta - 5^\circ\, 50')},$$

where the brackets indicate the common logarithm of a factor, the equation expressing the condition of minimum distance of centers, takes the form

$$Q - E = 14^\circ\, 42'\, 43''.3\,.$$

In applying these equations to the solution of the problem, we proceed by successive approximations; if no nearer value is at hand, we may take the time of the occurrence of the phenomenon at the center of the earth as a first approximation. We then compute Q and E for the assumed time and the given place. If then the equation

$$Q - E = 14^\circ\, 42'\, 43''.3$$

is satisfied, the assumed time is correct; but if not, the error should be divided by an approximate value of the rate at which the function $Q - E$ is increasing, which may be taken equal to the rate of increase of Q for the center of the earth. This is $-\,1025''$ per minute. The assumed time being corrected by the addition of the quotient, the computation may be repeated. This process may be continued until a sufficiently exact time is obtained, with which may be found the exact values of D and Q.

Take, as an example, the finding of the time of least distance of centers at Madras; for which

$$\varphi = +\,13^\circ\, 4'.2, \qquad \omega = 202^\circ\, 42'.6,$$

whence for this place

$$\begin{aligned} \varDelta \sin Q &= x - [9.9886] \sin \vartheta\,, \\ \varDelta \cos Q &= y - 0.2070 - [9.5787] \cos \vartheta\,, \\ \tan E &= \frac{[8.0589] \sin(\vartheta - 34^\circ\, 1')}{1 + [8.3896] \cos(\vartheta - 5^\circ\, 50')}\,. \end{aligned}$$

Assume $11^{h}\, 4^{m}.6$ as an approximate value of the time; for which

$$\begin{array}{lll} \mu = \,168^\circ\, 3'.8\,, & \vartheta = -\,34^\circ\, 38'.8\,, & x = +\,7.6652\,, \\ y = +\,32.9566\,, & \varDelta \sin Q = +\,8.2190\,, & \varDelta \cos Q = +\,32.4378\,, \\ Q = \,14^\circ\, 13'\, 5''.6\,, & E = -\,36'\, 0''.6\,, & Q - E = \,14^\circ\, 49'\, 6''.2\,. \end{array}$$

The error is, then, $-6' \, 22''.9$, and the correction to the assumed time,

$$\frac{-382''.9}{-1025''} \times 1^{\mathrm{m}} = +\,0^{\mathrm{m}}.3734\,.$$

If the computation be repeated for the time $11^{\mathrm{h}}\ 4^{\mathrm{m}}.9734$, the error of the value of $Q - E$ will be found to be only $13''$. Regarding this result as sufficiently accurate, we compute, for this time, Q and D, and find

$$Q = 14^{\circ}\, 6'\, 32'', \quad D = 819''.42 = 13'\, 39''.42\,.$$

These distances and angles of position are, it must be remembered, actual, not apparent. To obtain the last, the effect of refraction would have to be considered.

LOCALITIES FAVORABLE FOR THE DETERMINATION OF PARALLAX.

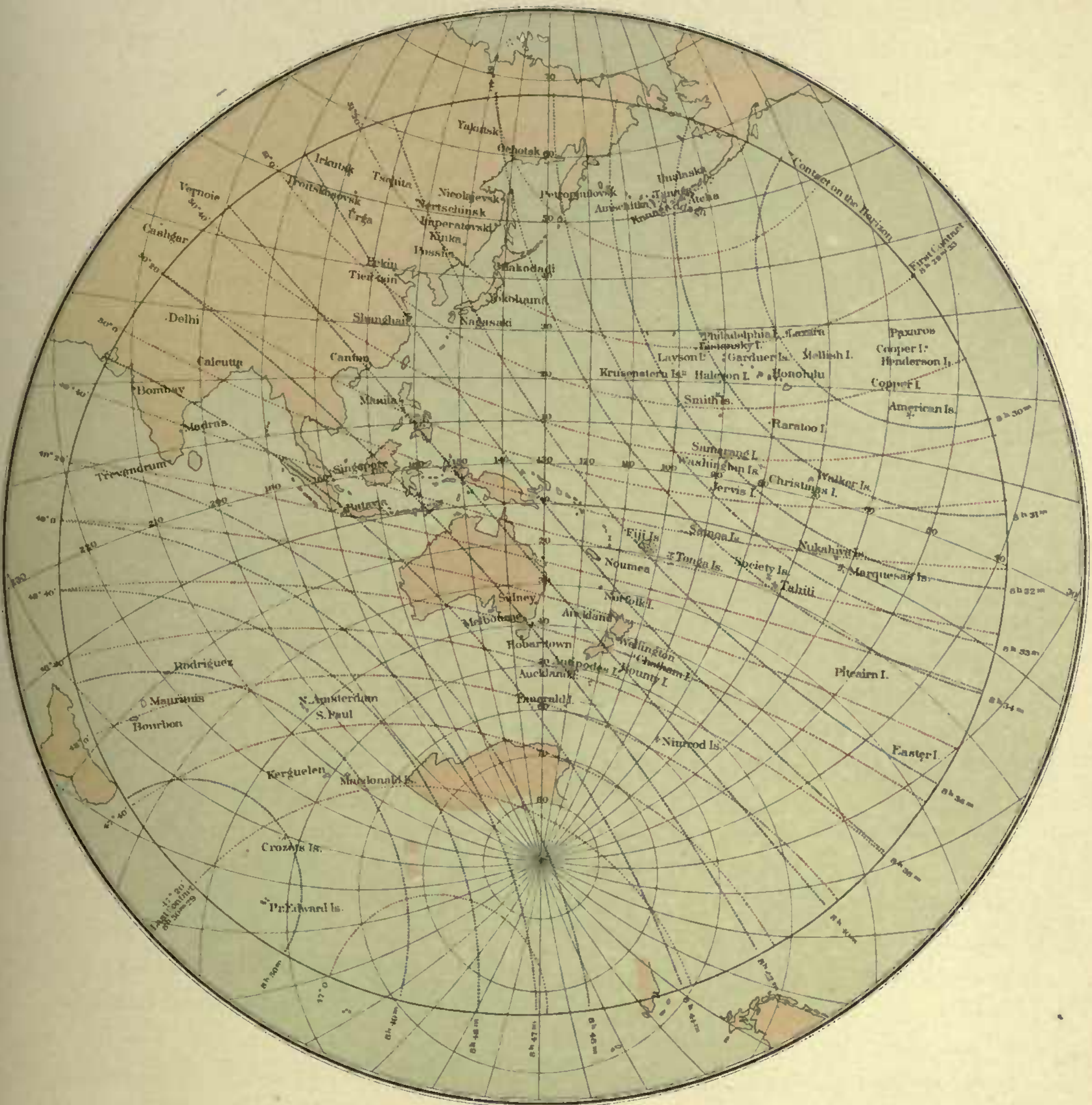
A list of localities favorably situated for observations of the contacts, with a view to the determination of the parallax, may be given in a few words.

For the ingress accelerated by parallax, we have, in the first place, the Hawaiian Islands; next, the most southerly and westerly of the Aleutian Islands, the southern part of Kamchatka, and Japan, especially the northern islands; also the Marquesas Islands, and, if more stations are desired, perhaps in the long series of islands stretching west-northwest from the Hawaiian Islands some might be found available. We may mention the small islands lying between the Hawaiian and Marquesas Islands.

For the ingress retarded by parallax, we have the islands of Saint Paul, New Amsterdam, Kerguelen, Bourbon, Mauritius, Diego Rodriguez, Crozet, Prince Edward, and Madagascar, where, however, only the interior contact will be visible, and on the eastern coast at an altitude from 5° to 6°.

For the egress accelerated by parallax, we have New Zealand and the small islands to the southward and eastward. With respect to the latter, we may note that on some maps may be found a group of small islands, called the Nimrod Islands, and placed in longitude 80° west from Washington and in latitude 57° south. Here the interior contact occurs at an altitude of 9°, and if these islands are of a sufficient size for the establishment of an observing station on them, it would be a tolerably good one, as far as geographical position is concerned. To these we may add Norfolk Island, New Caledonia, the Fiji Islands, Van Diemen's Land, and the southeastern part of Australia.

For the egress retarded by parallax, Southwestern Siberia, the region immediately east of the Caspian Sea, Persia, the Caucasus, Asia Minor, Syria, Arabia, and Egypt contain the best stations.



TRANSIT OF VENUS, DEC. 8, 1874.

CHART NO. I. INGRESS, EXTERIOR CONTACT

Scale for altitudes

0° 5° 10° 15° 20° 25° 30°

LEGEND

The broken lines in blue are for synchronism of contact.
The broken lines in red are for contact at the same point of the solar disk.

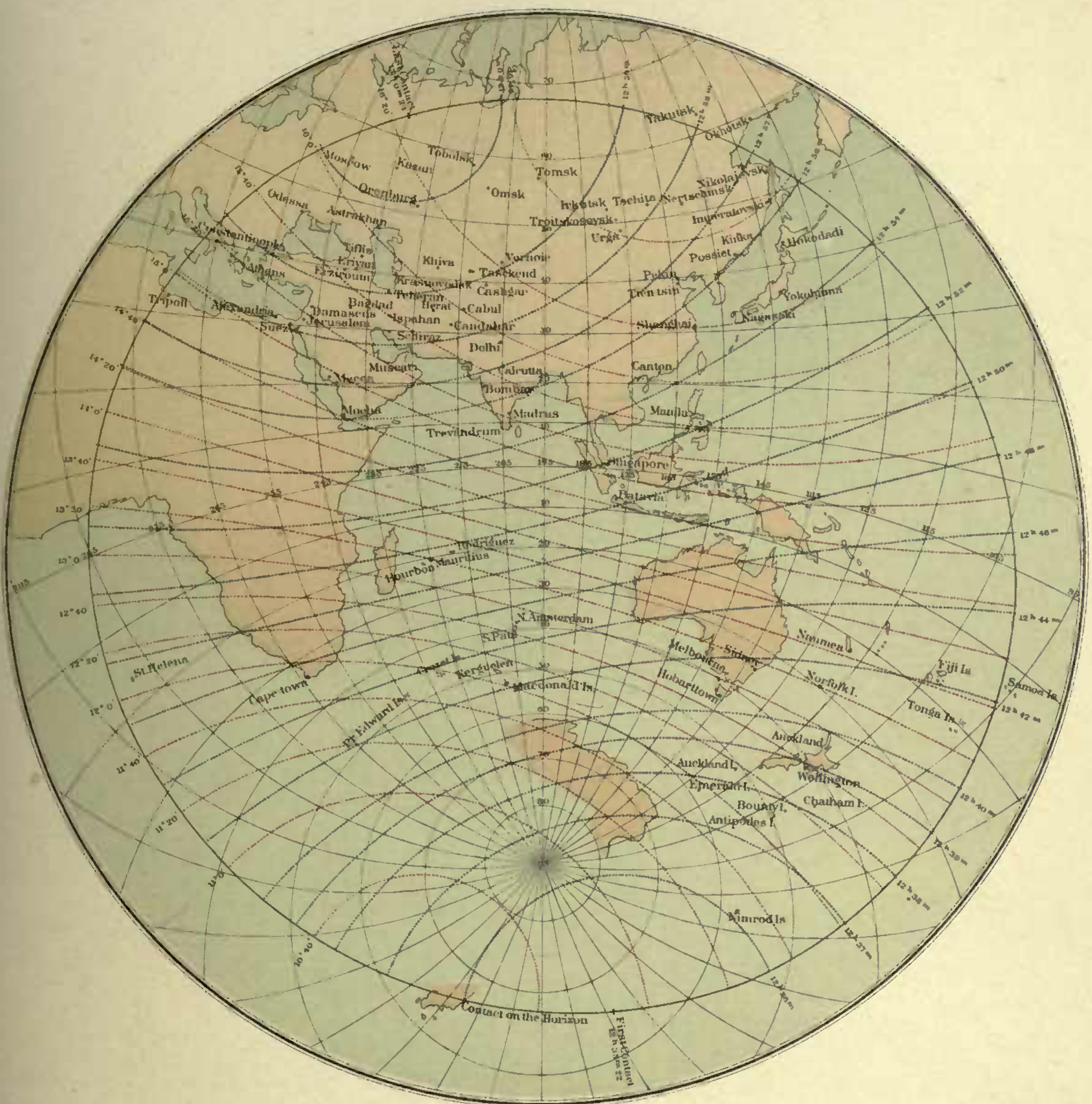
TRANSIT OF VENUS, DEC. 8, 1874.

CHART NO. 2. INGRESS, INTERIOR CONTACT

Scale for altitudes
0° 5° 10° 15° 20° 25° 30°

LEGEND

The broken lines in blue are for synchronism of contact.
The broken lines in red are for contact at the same point of the solar disk.



TRANSIT OF VENUS, DEC. 8, 1874

CHART NO. 3. EGRESS, INTERIOR CONTACT

Scale for altitudes
0° 5° 10° 15° 20° 25° 30°

LEGEND

The broken lines in blue are for synchronism of contact.
The broken lines in red are for contact at the same point of the solar disk.

TRANSIT OF VENUS, DEC. 8, 1874.

CHART NO. 4. EGRESS, EXTERIOR CONTACT

Scale for altitudes

0° 5° 10° 15° 20° 25° 30°

LEGEND

The broken lines in blue are for synchronism of contact.
The broken lines in red are for contact at the same point of the solar disk.

MEMOIR No. 14.

A Method of Computing Absolute Perturbations.

(Astronomische Nachrichten, Vol. 83, pp. 209-224, 1874.)

The object of this article is to call the attention of astronomers to the notable abbreviations which are produced in some parts of the formulas for perturbations by the introduction of the true anomaly as the variable according to which the integrations are to be executed. Prof. Hansen, in his later disquisitions, has substituted the eccentric anomaly as the independent variable in place of the mean anomaly, or what is the same thing, the time; and he regards this step as constituting a remarkable amelioration of the method. The method here explained will, as far as coordinates are concerned, be the same as that of Laplace, but the same use will be made of the true anomaly in the elliptic orbit, as independent variable, as that which Hansen has made of the eccentric anomaly.

The following notation and equations are so familiar that they seem to need no explanation:

$$\left.\begin{aligned} R &= m'\left(\frac{1}{\Delta}-\frac{r\cos\psi}{r'^2}\right)+m''\left(\frac{1}{\Delta_1}-\frac{r\cos\psi_1}{r''^2}\right)+\ldots,\\ \frac{d^2x}{dt^2}+\frac{\mu}{r^3}x &= \frac{\partial R}{\partial x},\\ \frac{d^2y}{dt^2}+\frac{\mu}{r^3}y &= \frac{\partial R}{\partial y},\\ \frac{d^2z}{dt^2}+\frac{\mu}{r^3}z &= \frac{\partial R}{\partial z}. \end{aligned}\right\} \tag{1}$$

Let us now suppose that each coordinate of the disturbed planet is separated into two parts, such that

$$x=x_0+\delta x,\quad y=y_0+\delta y,\quad z=z_0+\delta z,$$

the first of which, x_0, y_0, z_0 satisfy the differential equations

$$\frac{d^2x_0}{dt^2}+\frac{\mu}{r_0^3}x_0=0,\quad \frac{d^2y_0}{dt^2}+\frac{\mu}{r_0^3}y_0=0,\quad \frac{d^2z_0}{dt^2}+\frac{\mu}{r_0^3}z_0=0,$$

where $r_0^2=x_0^2+y_0^2+z_0^2$, and the second, δx, δy, δz are of the order of the disturbing forces.

It is evident that this separation is, to a certain extent, arbitrary, as certain functions of t might be added to x_0, y_0, z_0 without their ceasing to satisfy the differential equations determining them, and then δx, δy, δz would necessarily be diminished by the same functions. This indetermination is eliminated in different ways according to the circumstances attending the computation of the perturbations.

If x_0, y_0, z_0 are derived from the elements osculating for a certain epoch, it is plain that δx, δy, δz ought to vanish at this epoch, as also their first differentials with respect to the time. This will be accomplished by taking all the integrations, which δx, δy, δz involve, between the limits $t = 0$ and $t = t$. If the perturbations are computed from so called mean elements, the six arbitrary constants which δx, δy, δz involve, must be determined in accordance with the suppositions upon which the mean elements have been derived.

We will now write

$$r = r_0 + \delta r,$$

$$dR = \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz,$$

$$r \frac{\partial R}{\partial r} = x \frac{\partial R}{\partial x} + y \frac{\partial R}{\partial y} + z \frac{\partial R}{\partial z},$$

dR is then the differential of R when the coordinates of the disturbed planet alone vary. The last equation is evidently correct, when, in the first member, we suppose R to be expressed in terms of r and two other coordinates which make $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$ independent of r.

By multiplying the equations which determine x, y, z, severally by $2dx$, $2dy$, $2dz$, adding the products and integrating,

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2\mu}{r} + \frac{\mu}{a} = 2 \int dR, \tag{2}$$

where $\frac{\mu}{a}$ is the constant added to complete the integral, and we suppose that it is such that the equation

$$\frac{dx_0^2 + dy_0^2 + dz_0^2}{dt^2} - \frac{2\mu}{r_0} + \frac{\mu}{a} = 0$$

is satisfied; if there is any residual constant part, it must be supposed contained in the term $2\int dR$. By multiplying the differential equations deter-

mining x, y, z, severally by these quantities and adding the products to equation (2), we get

$$\tfrac{1}{2}\frac{d^2r^2}{dt^2} - \frac{\mu}{r} + \frac{\mu}{a} = 2\int dR + r\frac{\partial R}{\partial r}.$$

By using the equation $r = r_0 + \delta r$, this can be readily transformed into

$$\frac{d^2(r_0\delta r)}{dt^2} + \frac{\mu}{r_0} r_0\delta r = 2\int dR + r\frac{\partial R}{\partial r} - \tfrac{1}{2}\frac{d^2(\delta r)^2}{dt^2} + \frac{\mu(\delta r)^2}{r_0^3 r}.$$

In like manner equations (1) can be transformed into

$$\frac{d^2\delta x}{dt^2} + \frac{\mu}{r_0^3}\delta x = \frac{\partial R}{\partial x} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu x,$$

$$\frac{d^2\delta y}{dt^2} + \frac{\mu}{r_0^3}\delta y = \frac{\partial R}{\partial y} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu y$$

$$\frac{d^2\delta z}{dt^2} + \frac{\mu}{r_0^3}\delta z = \frac{\partial R}{\partial z} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu z.$$

For the sake of brevity put

$$\left.\begin{aligned} Q_r &= 2\int dR + r\frac{\partial R}{\partial r} - \tfrac{1}{2}\frac{d^2(\delta r)^2}{dt^2} + \frac{\mu(\delta r)^2}{r_0^3 r}, \\ Q_x &= \frac{\partial R}{\partial x} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu x, \\ Q_y &= \frac{\partial R}{\partial y} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu y, \\ Q_z &= \frac{\partial R}{\partial z} + \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\mu z. \end{aligned}\right\} \qquad (3)$$

Then our differential equations take the form

$$\left.\begin{aligned} \frac{d^2(r_0\delta r)}{dt^2} + \frac{\mu}{r_0^3} r_0\delta r &= Q_r, \\ \frac{d^2\delta x}{dt^2} + \frac{\mu}{r_0^3}\delta x &= Q_x, \\ \frac{d^2\delta y}{dt^2} + \frac{\mu}{r_0^3}\delta y &= Q_y, \\ \frac{d^2\delta z}{dt^2} + \frac{\mu}{r_0^3}\delta z &= Q_z, \end{aligned}\right\} \qquad (4)$$

The problem of elliptic motion being supposed completely solved, $\frac{\mu}{r_0^3}$ is a known function of t, and

$$\frac{d^2q}{dt^2} + \frac{\mu}{r_0^3} q = 0, \qquad (5)$$

is a linear differential equation. According to the theory of this class of differential equations, the value of q has the form

$$q = K_1q_1 + K_2q_2,$$

K_1 and K_2 being the arbitrary constants and q_1 and q_2 any particular solutions independent of each other. Then there must necessarily exist the two equations

$$\frac{d^2q_1}{dt^2} + \frac{\mu}{r_0^3}\,q_1 = 0\,, \quad \frac{d^2q_2}{dt^2} + \frac{\mu}{r_0^3}\,q_2 = 0\,. \tag{6}$$

By the elimination of $\frac{\mu}{r_0^3}$ from these is obtained

$$\frac{q_1d^2q_2 - q_2d^2q_1}{dt^2} = 0\,.$$

This is an exact differential and integrating

$$\frac{q_1dq_2 - q_2dq_1}{dt} = \text{a constant}\,. \tag{7}$$

This constant is arbitrary and may be taken at will; for the sake of simplicity, we assume it equal to unity.

Taking now the more general equation

$$\frac{d^2q}{dt^2} + \frac{\mu}{r_0^3}\,q = Q\,,$$

let us eliminate $\frac{\mu}{r_0^3}$ from this and equations (6). We get

$$\frac{q_1d^2q - qd^2q_1}{dt^2} = Qq_1\,,$$

$$\frac{q_2d^2q - qd^2q_2}{dt^2} = Qq_2\,,$$

and, taking the integrals,

$$\frac{q_1dq - qdq_1}{dt} = \quad K_2 + \int q_1Qdt\,,$$

$$\frac{q_2dq - qdq_2}{dt} = -K_1 + \int q_2Qdt\,.$$

Whence is obtained, regard being had to equation (7),

$$q = K_1q_1 + K_2q_2 + q_2\int q_1Qdt - q_1\int q_2Qdt\,.$$

The constants K_1 and K_2 may be regarded as contained in the integrals $\int q_2\, Qdt$ and $\int q_1\, Qdt$. Applying these results to equations (4), there result

$$\left.\begin{aligned}
r_0\delta r &= q_2\int q_1 Q_r dt - q_1\int q_2 Q_r dt,\\
\delta x &= q_2\int q_1 Q_x dt - q_1\int q_2 Q_x dt,\\
\delta y &= q_2\int q_1 Q_y dt - q_1\int q_2 Q_y dt,\\
\delta z &= q_2\int q_1 Q_z dt - q_1\int q_2 Q_z dt.
\end{aligned}\right\} \tag{8}$$

These equations must satisfy the relation

$$r_0\delta r = x_0\delta x + y_0\delta y + z_0\delta z + \tfrac{1}{2}\left[\delta x^2 + \delta y^2 + \delta z^2 - \delta r^2\right]. \tag{9}$$

It is, however, necessary to employ all of equations (8), since, in proceeding by successive approximations, as we are obliged to, we cannot get the values of Q_x, Q_y, Q_z, until δr is known. These equations contain, in the general case, nine arbitrary constants, viz., the one added to the term $2\int dR$ in Q_r, and the eight introduced by the eight integrals of equations (8). But the last will be reduced to six, independent of each other, by the condition (9), and the constant annexed to $\int dR$ will be determined in function of these six, by the condition

$$\frac{dx_0}{dt}\frac{d\delta x}{dt} + \frac{dy_0}{dt}\frac{d\delta y}{dt} + \frac{dz_0}{dt}\frac{d\delta z}{dt} + \frac{\mu}{r_0^2 r} r_0\delta r = \int dR - \tfrac{1}{2}\left[\left(\frac{d\delta x}{dt}\right)^2 + \left(\frac{d\delta y}{dt}\right)^2 + \left(\frac{d\delta z}{dt}\right)^2\right].$$

In the case, mentioned above, of osculating elements, all the constants are determined by making each integral expression vanish with t.

There is a remarkable procedure for reducing the right members of equations (8) to contain a single integral expression, which is due to Prof. Hansen. The factors q_1, q_2, outside the signs of integration, may evidently be removed within, if it is agreed to regard the t they contain, as constant in the integration. As it is necessary to keep this t distinct from the t of the quantities already under the sign of integration, we may write τ for it, and, to denote that any quantity, which is a function of t, has its t changed into τ, we will write ($^-$) above it. Thus making

$$N = \bar{q}_2 q_1 - \bar{q}_1 q_2, \tag{10}$$

we have the very simple expressions

$$r_0\delta r = \int N Q_r d\tau, \quad \delta x = \int N Q_x d\tau, \quad \delta y = \int N Q_y d\tau, \quad \delta z = \int N Q_z d\tau. \tag{11}$$

After the integration is finished, τ will be replaced by t. Since τ is regarded as constant, an arbitrary function of τ must be added to each of these expressions, which, after τ is changed into t, becomes an arbitrary function of t. These must, in each case, be so determined that the expressions (11) may coincide with (8). Any consideration of these arbitrary functions will be rendered unnecessary, by agreeing to take the integrations between limits, the upper of which is t itself, and the lower may be any constant. In the general case, then, an arbitrary expression of the form $K_1q_1 + K_2q_2$ must be added to each equation. In the case of osculating elements, mentioned above, if the lower limit is taken at zero, this arbitrary expression vanishes.

Equations (11) may be exhibited in the form of definite integrals, thus

$$r_0\delta r = -\int_0^t NQ_r d\tau, \quad \delta x = -\int_0^t NQ_x d\tau, \quad \delta y = -\int_0^t NQ_y d\tau, \quad \delta z = -\int_0^t NQ_z d\tau.$$

N may be regarded as an integrating factor whose value is virtually zero, but a part of the time, involved in its expression, is regarded as constant in the integration.

The values of q_1 and q_2 must now be determined. If

$$n = \sqrt{\frac{\mu}{a^3}}, \quad nt + c = \zeta = u - e \sin u,$$

where c and e are constants, and ζ and u respectively the mean and eccentric anomalies in the elliptic orbit of the disturbed planet, then

$$\frac{r_0}{a} = 1 - e \cos u, \quad d\zeta = \frac{r_0}{a} du.$$

Equation (5) becomes then

$$\frac{d^2q}{d\zeta^2} + \frac{a^3}{r_0^3} q = 0.$$

If u is made the independent variable, it becomes

$$(1 - e \cos u) \frac{d^2q}{du^2} - e \sin u \frac{dq}{du} + q = 0. \tag{12}$$

Differentiating this and removing the useless factor $1 - e \cos u$, we get

$$\frac{d^3q}{du^3} + \frac{dq}{du} = 0,$$

the integral of which is

$$q = K_1 \cos u + K_2 \sin u + K_3.$$

Determining K_3 so that equation (12) may be satisfied, $K_3 = -K_1 e$. Hence the complete integral of (12) is

$$q = K_1(\cos u - e) + K_2 \sin u .$$

It is evident now that we may take

$$q_1 = k(\cos u - e), \quad q_2 = k \sin u .$$

If these values are substituted in equation (7), it is found that $k^2 = \frac{1}{n}$; thus

$$q_1 = \sqrt{\frac{a^3 n}{\mu}}(\cos u - e) = \sqrt{\frac{an}{\mu}}\, r_0 \cos v ,$$

$$q_2 = \sqrt{\frac{a^3 n}{\mu}} \sin u \qquad = \sqrt{\frac{an}{\mu(1-e^2)}}\, r_0 \sin v ,$$

if v is the true anomaly of the disturbed planet in its elliptic orbit. Thus

$$N = \frac{a^3 n}{\mu}[\sin(\bar{u} - u) - e \sin \bar{u} + e \sin u]$$

$$= \frac{an}{\mu\sqrt{1-e^2}}\, \bar{r}_0 r_0 \sin(\bar{v} - v) .$$

We now change the independent variable t for the variable v. We have

$$dt = \frac{r_0^2 dv}{a^2 n \sqrt{1-e^2}} ,$$

whence

$$N dt = \frac{1}{\mu a(1-e^2)}\, \bar{r}_0 r_0^3 \sin(\bar{v} - v)\, dv .$$

Thus the expressions for the perturbations become

$$\left.\begin{aligned}
\delta r &= \frac{1}{\mu a(1-e^2)} \int Q_r r_0^3 \sin(\bar{v} - v)\, dv ,\\
\delta x &= \frac{r_0}{\mu a(1-e^2)} \int Q_x r_0^3 \sin(\bar{v} - v)\, dv ,\\
\delta y &= \frac{r_0}{\mu a(1-e^2)} \int Q_y r_0^3 \sin(\bar{v} - v)\, dv ,\\
\delta z &= \frac{r_0}{\mu a(1-e^2)} \int Q_z r_0^3 \sin(\bar{v} - v)\, dv .
\end{aligned}\right\} \qquad (13)$$

It will be perceived that, by this transformation, we have been enabled to get rid of the factor r_0 before δr, with a simplification of the right member of the equation.

These equations, although very symmetrical, present the inconvenience of being one too many. Hence, for the second and third, we substitute a single one. From the differential equations of motion,

$$\frac{xdy-ydx}{dt}=h+\int\left[x\frac{\partial R}{\partial y}-y\frac{\partial R}{\partial x}\right]dt,$$

where h is such a constant that

$$\frac{x_0dy_0-y_0dx_0}{dt}=h,$$

and i denoting the inclination of the plane of the elliptic orbit to the plane of xy,

$$h=\sqrt{\mu a(1-e^2)}\cos i.$$

Denoting by λ the longitude measured in the plane xy, so that $\tan\lambda=\frac{y}{x}$, and putting

$$Q_\lambda=x\frac{\partial R}{\partial y}-y\frac{\partial R}{\partial x}=\frac{\partial R}{\partial\lambda},$$

we shall have

$$(r^2-z^2)\frac{d\lambda}{dt}=h+\int Q_\lambda dt.$$

Supposing $\lambda=\lambda_0+\delta\lambda$, where $\tan\lambda_0=\frac{y_0}{x_0}$, the following equation is obtained for the determination of $\delta\lambda$:

$$\delta\lambda=\int\left[\int\frac{an}{\mu}Q_\lambda dt-\sqrt{1-e^2}\cos i\frac{(r+r_0)\delta r-(z+z_0)\delta z}{r_0^2-z_0^2}\right]\frac{a^2ndt}{r^2-z^2}.$$

Or, if v is made the independent variable, and for brevity we put $p=a(1-e^2)$, the expressions for the perturbations are

$$\left.\begin{aligned}\delta r&=\frac{1}{\mu p}\int Q_r r_0^3\sin(\bar v-v)\,dv,\\ \delta z&=\frac{r_0}{\mu p}\int Q_z r_0^3\sin(\bar v-v)\,dv,\\ \delta\lambda&=\int\left[\int\frac{r_0^2}{\mu p}Q_\lambda dv-\cos i\frac{(r+r_0)\delta r-(z+z_0)\delta z}{r_0^2-z_0^2}\right]\frac{r_0^2dv}{r^2-z^2}\end{aligned}\right\}\qquad(14)$$

These formulas are absolutely rigorous, since no terms have been neglected, and also perfectly general, as no restriction has been put upon the position of the plane of xy from which the coordinate z is measured. By adopting the plane of the elliptic orbit of the disturbed planet as the plane

of xy, the last equation is somewhat simplified. For then $i = 0$, and $z_0 = 0$, and $z = \delta z$; thus

$$\delta\lambda = \int\left[\int \frac{r_0^3}{\mu p} Q_\lambda dv - \frac{(r + r_0)\,\delta r - \delta z^2}{r_0^2}\right]\frac{r_0^2 dv}{r^2 - \delta z^2}. \tag{15}$$

Perturbations of the first order with respect to the disturbing forces.

Since, in this case, elliptic values are to be substituted for the coordinates in the functions Q_r, Q_z, Q_λ, there is no need any further of making a distinction between r_0 and r; hence the $(_0)$ will be omitted from the former. If we put

$$T = \frac{r^3}{\mu p} Q_r = \frac{r^3}{\mu p}\left[2\int dR + r\frac{\partial R}{\partial r}\right], \quad Z = \frac{r^3}{\mu p}\frac{\partial R}{\partial z}, \quad Y = \frac{r^3}{\mu p} Q_\lambda,$$

and $\delta\beta$ is the latitude of the disturbed planet measured from the plane of its elliptic orbit, and $\delta\lambda$ the perturbation of the longitude measured in this plane, our formulas, in this case, reduce to

$$\delta r = \int T \sin(\bar{v} - v)\, dv, \quad \delta\beta = \int Z \sin(\bar{v} - v)\, dv, \quad \delta\lambda = \int\left[\int Y dv - 2\frac{\delta r}{r}\right] dv.$$

Put now

$$X = \frac{r^4}{\mu p}\frac{\partial R}{\partial r},$$

then it will easily be found that

$$\frac{1}{\mu p} dR = r^{-2}\left[\frac{e \sin v}{p} X + Y\right] dv$$

Thus the shape, in which we shall employ our equations, is

$$\delta r = \int\left[X + 2r^3 \int r^{-2}\left(\frac{e \sin v}{p} X + Y\right) dv\right] \sin(\bar{v} - v)\, dv,$$

$$\delta\lambda = \int\left[\int Y dv - 2\frac{\partial r}{r}\right] dv,$$

$$\delta\beta = \int [Z \sin(\bar{v} - v)]\, dv.$$

The chief thing now to be done is to expand X, Y and Z in periodic series as functions of v. The elliptic values of the coordinates of the disturbed planet are readily expressed in terms of this variable, but the coordinates of the disturbing bodies will naturally be expressed in terms of their mean anomalies ζ', ζ'', etc. These last variables must be eliminated by means of the identities

$$\zeta' = \frac{n'}{n} v + c' - \frac{n'}{n} c - \frac{n'}{n}(v - \zeta),$$

$$\zeta'' = \frac{n''}{n} v + c'' - \frac{n''}{n} c - \frac{n''}{n}(v - \zeta),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Let us then put

$$\vartheta' = \frac{n'}{n} v + c' - \frac{n'}{n} c, \quad \vartheta'' = \frac{n''}{n} v + c'' - \frac{n''}{n} c, \text{ etc.,}$$

so that

$$\frac{d\vartheta'}{dv} = \frac{n'}{n}, \quad \frac{d\vartheta''}{dv} = \frac{n''}{n}, \text{ etc.}$$

Then ζ', ζ'', etc., will be replaced by the following values:

$$\zeta' = \vartheta' - \frac{n'}{n}(v - \zeta), \quad \zeta'' = \vartheta'' - \frac{n''}{n}(v - \zeta), \text{ etc.}$$

In the development of X, Y and Z in periodic series from particular values of these quantities, it will be better to make the differences of ϑ', ϑ'', etc., from v, the variables to be employed. Thus we shall put $w' = \vartheta' - v$, $w'' = \vartheta'' - v$, etc.

The formulas, to be written now, will be confined to the case of the action of one planet. The expressions for X, Y and Z are

$$X = \frac{m'}{h^2} r^4 \left[\frac{1}{\Delta^3} - \frac{1}{r'^3}\right] r' \cos\beta' \cos(\lambda' - \lambda) - \frac{m'}{h^2} \frac{r^5}{\Delta^3},$$

$$Y = \frac{m'}{h^2} r^3 \left[\frac{1}{\Delta^3} - \frac{1}{r'^3}\right] r' \cos\beta' \sin(\lambda' - \lambda),$$

$$Z = \frac{m'}{h^2} r^3 \left[\frac{1}{\Delta^3} - \frac{1}{r'^3}\right] r' \sin\beta',$$

where $h^2 = \mu a (1 - e^2)$, and

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos\beta' \cos(\lambda' - \lambda).$$

If the inclinations of the orbits of the two planets to some fixed plane, as the ecliptic, are denoted by i, i', and the longitudes of their ascending nodes by Ω, Ω', and the longitudes of their perihelia by π, π', we compute I, Θ, Θ', Π and Π' from

$$\begin{aligned}
\cos I &= \cos i \cos i' + \sin i \sin i' \cos(\Omega' - \Omega), \\
\sin I \cos(\theta - \Omega) &= -\sin i \cos i' + \cos i \sin i' \cos(\Omega' - \Omega), \\
\sin I \sin(\theta - \Omega) &= \sin i' \sin(\Omega' - \Omega) \\
\sin I \cos(\theta' - \Omega') &= \cos i \sin i' - \sin i \cos i' \cos(\Omega' - \Omega), \\
\sin I \sin(\theta' - \Omega') &= \sin i \sin(\Omega' - \Omega), \\
\Pi = \pi - \theta, &\quad \Pi' = \pi' - \theta'.
\end{aligned}$$

The circumference being divided into k equal parts with reference to v, compute for each of the k values of v, $0, \frac{2}{k}\pi, \frac{4}{k}\pi \ldots \frac{2(k-1)}{k}\pi$, the following quantities:

$$\tan\frac{u}{2}=\sqrt{\frac{1-e}{1+e}}\tan\frac{v}{2},$$

$$\zeta=u-e\sin u,$$

$$V=v-\frac{n'}{n}(v-\zeta),$$

$$r=\frac{p}{1+e\cos v},$$

$$K\cos(\Pi'-A)=\cos(v+\Pi),\qquad K'\cos(\Pi'-A')=\cos I\cos(v+\Pi),$$

$$K\sin(\Pi'-A)=\cos I\sin(v+\Pi),\qquad K'\sin(\Pi'-A')=\sin(v+\Pi),$$

$$G=2Kr,\quad G'=\frac{m'}{h^2}r^4K,\quad G''=\frac{m'}{h^2}r^5,\quad G'''=\frac{m'}{h^2}r^3K',\quad G''''=\frac{m'}{h^2}\sin I.r^3.$$

Several of these quantities, as u, ζ, V, r, will need to be computed only $\frac{k}{2}$ times, if k is a multiple of 2; and K, K', A, A' only $\frac{k}{4}$ times in the same case.

The circumference being divided into k' equal parts with reference to the variable w', compute for each of the kk' values of v and w' the following quantities, w' taking in succession the values 0, $\frac{2}{k'}\pi$, $\frac{4}{k'}\pi$ .... $\frac{2(k'-1)}{k'}\pi$:

$$\zeta'=V+w',\qquad u'-e'\sin u'=\zeta',$$

$$\sqrt{r'}\cos\frac{v'}{2}=\sqrt{a'(1-e')}\cos\frac{u'}{2},\qquad \sqrt{r'}\sin\frac{v}{2}=\sqrt{a'(1+e')}\sin\frac{u'}{2}.$$

If we have tables of the disturbing planet giving the true anomaly or the equation of the center and the radius vector or its logarithm with the argument mean anomaly, we can derive log r' and v' by means of their aid, and thus dispense with computing the last three equations. We now compute kk' times

$$\Delta^2=r^2+r'^2-Gr'\cos(v'+A),$$

$$X=G'\left[\frac{1}{\Delta^3}-\frac{1}{r'^3}\right]r'\cos(v'+A)-\frac{G''}{\Delta^3},$$

$$Y=G'''\left[\frac{1}{\Delta^3}-\frac{1}{r'^3}\right]r'\sin(v'+A'),$$

$$Z=G''''\left[\frac{1}{\Delta^3}-\frac{1}{r'^3}\right]r'\sin(v'+\Pi').$$

From these kk' special values of each of the quantities X, Y and Z, we deduce their developments in periodic series of the form

$$\Sigma_{ii'}\,[K_{ii'}^{(c)}\cos(iv-i'w')+K_{ii'}^{(s)}\sin(iv-i'w')].$$

This process is so well known that we need not here insert the formulas required for it; they will be found in Hansen's *Auseinandersetzung*, Part I, p. 159. A double application of these formulas will be necessary, the first relative to v, the second relative to w'. After these series are obtained, w' can be replaced by $\vartheta' - v$.

The series X is now to be multiplied by $\frac{e}{p}\sin v$, which, for every periodic term in X, will give two periodic terms, which will be added to Y. This result is next to be multiplied by

$$\left(\frac{1 + e\cos v}{p}\right)^2 = \frac{1 + \frac{1}{2}e^2}{p^2} + \frac{2e}{p^2}\cos v + \frac{e^2}{2p^2}\cos 2v.$$

There is now an integration to be effected. A table of logarithms of the integrating factors

$$[i, i'] = \frac{1}{i - i'\dfrac{n'}{n}}$$

will now be made for all combinations of i and i' which occur in the periodic series. If the last result contains a term

$$K\,{}^{\cos}_{\sin}\,(iv - i'\vartheta'),$$

the corresponding term of the integrated result will be

$$\pm\,[i, i']\,K\,{}^{\sin}_{\cos}\,(iv - i'\vartheta').$$

A multiplication by $2r^3$ is now to be made. We have

$$\frac{r^3}{a^3} = \tfrac{1}{2}E_0 - E_1\cos v + E_2\cos 2v - E_3\cos 3v + \ldots,$$

where the rigorous value of the coefficients is given by the equation

$$E_i = \sqrt{1 - e^2}\,[2 + e^2 + 3i\sqrt{1 - e^2} + i^2(1 - e^2)]\left(\frac{e}{1 + \sqrt{1 - e^2}}\right)^i.$$

This multiplication accomplished, the product is to be added to X. If this result has a term

$$K\,{}^{\cos}_{\sin}\,(iv - i'\vartheta'),$$

then δr has its corresponding term

$$-[i-1, i'][i+1, i']\,K\,{}^{\cos}_{\sin}\,(iv - i'\vartheta'),$$

except in the case where $i = 1$, $i' = 0$, when we have, instead of this,

$$\pm \tfrac{1}{2} Kv \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} v\,.$$

Having thus obtained δr, we multiply it by

$$\frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos v\,.$$

The result, which is the perturbation of the natural logarithm of r, must be doubled and then subtracted from $\int Y dv$. Another integration being executed on this result, we have $\delta\lambda$ the perturbation of the longitude measured in the plane of the fixed elliptic orbit.

Finally, $\delta\beta$ will be obtained by treating Z to the same kind of integration as that last used in obtaining δr; that is, in general, each coefficient of Z will be multiplied by the proper value of $-[i-1, i'][i+1, i']$ which corresponds to it.

Perturbations of the second order with respect to the disturbing forces.

Calling the parts of the perturbations of r, β, λ, which are of two dimensions with respect to the planetary masses, $\delta^2 r$, $\delta^2\beta$, $\delta^2\lambda$, so that we have, with errors of the third order,

$$r = r_0 + \delta r + \delta^2 r, \quad \beta = \delta\beta + \delta^2\beta, \quad \lambda = \lambda_0 + \delta\lambda + \delta^2\lambda,$$

where δr, $\delta\beta$, $\delta\lambda$ are the perturbations which have just been determined, we shall have

$$\delta^2 r = \int \frac{r^3}{h^2}\,\delta Q_r \sin(\bar{v} - v)\,dv\,,$$

$$\delta^2\beta = \int \frac{r^3}{h^2}\,\delta Q_z \sin(\bar{v} - v)\,dv - \frac{\delta r}{r}\,\delta\beta\,,$$

$$\delta^2\lambda = \int\left[\int \frac{r^3}{h^2}\,\delta Q_\lambda\,dv - 2\frac{\delta^2 r}{r} - \left(\frac{\delta r}{r}\right)^2 + \delta\beta^2 - 2\frac{\delta r}{r}\,\frac{d\,.\,\delta\lambda}{dv}\right]dv\,,$$

where, as before, there is no need of any distinction between r_0 and r. The following are the expressions for δQ_r and δQ_z,

$$\frac{r^3}{h^2}\,\delta Q_r = \frac{r^3}{h^2}\,\delta\left(r\frac{\partial R}{\partial r}\right) + 2\frac{r^3}{h^2}\int d\delta R - \frac{1}{2r}\,\frac{d^2(\delta r)^2}{dv^2} + \frac{e}{p}\sin v\,\frac{d(\delta r)^2}{dv} + \frac{1}{p}(\delta r)^2,$$

$$\frac{r^3}{h^2}\,\delta Q_z = \frac{r^3}{h^2}\,\delta\left(\frac{\partial R}{\partial z}\right) + \frac{3}{p}\,\delta r\delta\beta\,.$$

Bearing in mind that X, Y, Z are homogeneous functions of r and r', it will be easy to deduce the following equations:

$$\frac{r^3}{h^2}\delta\left(r\frac{\partial R}{\partial r}\right)=\left(r\frac{\partial X}{\partial r}-3X\right)\frac{\delta r}{r}-\left(r\frac{\partial X}{\partial r}-2X\right)\frac{\delta r'}{r'}$$
$$+\frac{\partial X}{\partial \lambda}(\delta\lambda-\delta\lambda')+r\left(r\frac{\partial Z}{\partial r}-3Z\right)\delta\beta+\frac{\partial X}{\partial \beta'}\delta\beta',$$

$$\frac{r^2}{h^2}\delta Q_\lambda \quad =\frac{1}{r}\frac{\partial X}{\partial \lambda}\frac{\delta r}{r}-\left(\frac{1}{r}\frac{\partial X}{\partial \lambda}+Y\right)\frac{\delta r'}{r'}+\frac{\partial Y}{\partial \lambda}(\delta\lambda-\delta\lambda')+\frac{\partial Z}{\partial \lambda}\delta\beta+\frac{\partial Y}{\partial \beta'}\delta\beta',$$

$$\frac{r^3}{h^2}\delta\left(\frac{\partial R}{\partial z}\right)=\left(r\frac{\partial Z}{\partial r}-3Z\right)\frac{\delta r}{r}-\left(r\frac{\partial Z}{\partial r}-Z\right)\frac{\delta r'}{r'}+\frac{\partial Z}{\partial \lambda}(\delta\lambda-\delta\lambda')+\frac{\partial Z}{\partial \beta}\delta\beta+\frac{\partial Z}{\partial \beta'}\delta\beta',$$

$$2\frac{r^3}{h^2}\int d\delta R=2r^3\int r^{-3}\left[\frac{X}{r}\frac{d.\frac{\delta r}{r}}{dv}+Y\frac{d.\delta\lambda}{dv}+Z\frac{d.\delta\beta}{dv}+\frac{e}{p}\sin v.\frac{r^3}{h^2}\delta\left(r\frac{\partial R}{\partial r}\right)+\frac{r^2}{h^2}\delta Q_\lambda\right]dv,$$

where the differential coefficients $\dfrac{d.\frac{\delta r}{r}}{dv}$, $\dfrac{d.\delta\lambda}{dv}$, $\dfrac{d.\delta\beta}{dv}$ are complete with respect to the independent variable v.

In computing the values of these functions, $\dfrac{\delta r'}{r'}$, $\delta\lambda'$ and $\delta\beta'$ must be expressed as functions of v. Hence, if they are at first expressed in terms of t, it must be eliminated by means of the equation

$$nt+c=v-E_1\sin v+\tfrac{1}{2}E_2\sin 2v-\tfrac{1}{3}E_3\sin 3v+\ldots,$$

where the rigorous value of E_i is

$$E_i=2\left[1+i\sqrt{1-e^2}\right]\left(\frac{e}{1+\sqrt{1-e^2}}\right)^i.$$

We may have given only the perturbation of the orbit longitude and the latitude above the elliptic orbit of the disturbing planet; in this case, calling the latter $\delta\eta'$, the values of $\delta\lambda'$ and $\delta\beta'$ will be given by the equations

$$\delta\lambda'=\frac{\cos I}{\cos^2\beta'}\delta v'-\frac{\sin I\cos(v'+\Pi')}{\cos^2\beta'}\delta\eta',$$
$$\delta\beta'=\frac{\cos I}{\cos\beta'}\delta\eta'+\frac{\sin I\cos(v'+\Pi')}{\cos\beta'}\delta v'.$$

We see that, in order to obtain the perturbations of the second order it will be necessary to have, expressed in periodic series in terms of v, the following nine quantities:

$$r\frac{\partial X}{\partial r},\quad r\frac{\partial Z}{\partial r},\quad \frac{\partial Z}{\partial \beta},\quad \frac{\partial X}{\partial \lambda},\quad \frac{\partial Y}{\partial \lambda},\quad \frac{\partial Z}{\partial \lambda},\quad \frac{\partial X}{\partial \beta'},\quad \frac{\partial Y}{\partial \beta'},\quad \frac{\partial Z}{\partial \beta'}.$$

For six of these whose expressions are

$$r\frac{\partial X}{\partial r}=4X-\frac{m'}{h^2}\frac{r^5}{\Delta^3}+3\frac{m'}{h^2}r^3\left(\frac{(r'^2-r^2)^2}{4\Delta^5}-\frac{r'^2-r^2}{2\Delta^3}+\frac{1}{4\Delta}\right),$$

$$r\frac{\partial Z}{\partial r}=3Z+\frac{3}{2}\frac{m'}{h^2}r^3\left(\frac{r'^2-r^2}{\Delta^5}-\frac{1}{\Delta^3}\right)r'\sin\beta',$$

$$\frac{\partial Z}{\partial \beta}=\frac{m'}{h^2}r^4\left(\frac{3r'^2\sin^2\beta'}{\Delta^5}-\frac{1}{\Delta^3}\right),$$

$$\frac{\partial X}{\partial \beta'}=\frac{m'}{h^2}r^4\left(\frac{3}{2}\frac{r^2-r'^2}{\Delta^5}+\frac{1}{2}\frac{1}{\Delta^3}+\frac{1}{r'^3}\right)r'\sin\beta'\cos(\lambda'-\lambda),$$

$$\frac{\partial Y}{\partial \beta'}=-\frac{m'}{h^2}r^3\left(\frac{3}{2}\frac{r^2+r'^2}{\Delta^5}-\frac{1}{2}\frac{1}{\Delta^3}-\frac{1}{r'^3}\right)r'\sin\beta'\sin(\lambda'-\lambda),$$

$$\frac{\partial Z}{\partial \beta'}=\frac{m'}{h^2}r^3\left[\left(\frac{1}{\Delta^3}-\frac{1}{r'^3}\right)r'\cos\beta'-\frac{3rr'^2\sin^2\beta'\cos(\lambda'-\lambda)}{\Delta^5}\right],$$

the same method must be used as that which has been given for X, Y, Z. The remaining three, X, Y and Z being considered as functions of the two variables v and ϑ', can be obtained from the equations

$$\frac{\partial X}{\partial \lambda}=\frac{\partial X}{\partial v}+\frac{n'}{n}\left(1-\frac{r^2}{a^2\sqrt{1-e^2}}\right)\frac{\partial X}{\partial \vartheta'}-\frac{e}{p}\sin v\,.\,r^2\frac{\partial X}{\partial r},$$

$$\frac{\partial Y}{\partial \lambda}=\frac{\partial Y}{\partial v}+\frac{n'}{n}\left(1-\frac{r^2}{a^2\sqrt{1-e^2}}\right)\frac{\partial Y}{\partial \vartheta'}-\frac{e}{p}\sin v\,.\,r^2\frac{\partial Y}{\partial r},$$

$$\frac{\partial Z}{\partial \lambda}=\frac{\partial Z}{\partial v}+\frac{n'}{n}\left(1-\frac{r^2}{a^2\sqrt{1-c^2}}\right)\frac{\partial Z}{\partial \vartheta'}-\frac{e}{p}\sin v\,.\,r^2\frac{\partial Z}{\partial r}.$$

The factor

$$1-\frac{r^2}{a^2\sqrt{1-e^2}}=E_1\cos v-E_2\cos 2v+E_3\cos 3v-\ldots,$$

where

$$E_i=2[1+i\sqrt{1-e^2}]\left(\frac{e}{1+\sqrt{1-e^2}}\right)^i.$$

Moreover, we have the relation

$$r^2\frac{\partial Y}{\partial r}=\frac{\partial X}{\partial \lambda}+2rY.$$

The factor r is given by the equation

$$\frac{r}{a}=\tfrac{1}{2}E_0-E_1\cos v+E_2\cos 2v-E_3\cos 3v+\ldots,$$

where

$$E_i=2\sqrt{1-e^2}\left(\frac{e}{1+\sqrt{1-e^2}}\right)^i.$$

The values of $\lambda' - \lambda$ and β', necessary for the computation of the first six quantities, can be obtained from the equations

$$\begin{aligned}\cos\beta' \cos(\lambda' - \lambda) &= K\cos(v' + A)\,,\\ \cos\beta' \sin(\lambda' - \lambda) &= K' \sin(v' + A')\,,\\ \sin\beta' &= \sin I \sin(v' + \Pi')\,.\end{aligned}$$

The terms to be integrated in the second approximation have the general form

$$(C + C'v)\, {\sin \atop \cos}(iv - i'\vartheta' - i''\vartheta'')\,.$$

If these terms are integrated with respect to v, we have

$$\mp [i, i', i''](C + C'v)\, {\cos \atop \sin}(iv - i'\vartheta' - i''\vartheta'') + [i, i', i'']^2\, C' {\sin \atop \cos}(iv - i'\vartheta' - i''\vartheta'')\,,$$

where

$$[i, i', i''] = \frac{1}{i - i'\dfrac{n'}{n} - i''\dfrac{n''}{n}}\,.$$

If they are integrated after having been multiplied by the factor $\sin(\bar{v} - v)$, the result is

$$-[i-1, i', i''][i+1, i', i''](C + C'v)\, {\sin \atop \cos}(iv - i'\vartheta' - i''\vartheta'')$$
$$\mp \frac{2\,[i-1, i', i'']^2\,[i+1, i', i'']^2}{[i, i', i'']}\, C' {\cos \atop \sin}(iv - i'\vartheta' - i''\vartheta''),$$

except in the case where $i = 1$, $i' = 0$, $i'' = 0$, when we shall have

$$\pm\left(\tfrac{1}{8}C' - \tfrac{1}{2}Cv - \tfrac{1}{4}C'v^2\right) {\cos \atop \sin} v + \tfrac{1}{4}(C + C'v)\, {\sin \atop \cos} v\,.$$

The labor of computing perturbations of the second order is, in some sort, measured by the number of multiplications to be made of two periodic series, each involving double arguments. In this method, in the case of one disturbing planet, there are 22, or 25 if one thinks that the multiplications involving $\delta\lambda'$ ought to be considered as distinct from those involving $\delta\lambda$. If all the terms involving $\sin I$ as a factor be neglected, the number of these multiplications is diminished by 12.

It is my intention to illustrate this method by applying it to the computation of the perturbations of the first order of Ceres by Jupiter.

MEMOIR No. 15.

On a Long Period Inequality in the Motion of Hestia Arising from the Action of the Earth.

(Astronomische Nachrichten, Vol. LXXXIV, pp. 41-44, 1874.)

While the attention of all is directed to the more exact determination of the constant of solar parallax from the approaching transit of Venus, it may be of interest to notice another source from which, at least in the future, can be obtained the value of this constant.

Several of the asteroids have periods of revolution approximating quite closely to four years; hence, in their longitudes are long period equations of the form

$$k \sin [4g - g' + K],$$

g and g' being the mean anomalies of the asteroid and the earth. Should k be quite large, after the inequality has run through a considerable portion of its period, we can, from this source, determine a pretty exact value of the earth's mass, and thence, by the known formula, the corresponding value of the constant of solar parallax.

In order to see what may be expected in this direction, I have computed this inequality, as far as the first power of the disturbing force is concerned, for Hestia. This asteroid has been selected on account of its large eccentricity and the near approach of its period to four years. The elements employed (as many as we have need of), from the *Berliner Jahrbuch* for 1875, and from Leverrier's *Annales de l'Observatoire*, Tome IV, are as follows:

HESTIA. *Osculating*, 1865, *July* 26.		THE EARTH. *Mean Elements for the same epoch.*
$\pi = 354° \; 14' \; 18''.7$	M. E. 1870.0	$\pi' = 100° \; 41' \; 25''.0$
$\varphi = \; 9 \; 26 \; 55.8$	M. E. 1870.0	$\varphi' = \; 0 \; 57 \; 38.1$
$\Omega = 181 \; 30 \; 35.3$	M. E. 1870.0	$\mu' = 3548''.19286$
$i = \; 2 \; 17 \; 30.0$	M. E. 1870.0	$m' = \frac{1}{322800}$
$\mu = 883''.56391$		
$\log a = 0.4025124$		

These elements give $\mu' - 4\mu = 13''.93722$, whence the period of the inequality, in this case, is 254.6 years.

By a quite rigorous process, similar to that employed in Hansen's *Auseinandersetzung*, the terms of $\frac{a}{\Delta}$ depending on the argument $4g - g'$ have been found to be

$$-0.00174923 \cos(4g - g') + 0.01104188 \sin(4g - g').$$

And, in like manner, the second part of the disturbing function $-\frac{ar}{r'^2}\cos\psi$ contains the terms

$$+0.00257586 \cos(4g - g') - 0.00872291 \sin(4g - g').$$

Thus aR contains the terms

$$+0.00082663 \cos(4g - g') + 0.00231897 \sin(4g - g').$$

Multiplying these by the factor

$$-\frac{12\mu^2 m'}{(4\mu - \mu')^2} \times 206264''.8,$$

we have the inequality sought,

$$\int ndt = 75''.869 \sin(4g - g' + 109^\circ\ 37'\ 10'').$$

The effect of this inequality on the geocentric position of Hestia at opposition is got, somewhat roughly, by multiplying the preceding expression by $\frac{a}{a-1}$, and hence, at a maximum, may amount to about $125''$.

It must be confessed that the determination of the earth's mass from this source is attended with the inconvenience of having to compute very accurately the perturbations of Hestia by Jupiter; and among these is a very large inequality having the argument $g - 3g''$, whose period is nearly the same as that of the inequality just determined. Hence it will be necessary to proceed with a very accurate value of Jupiter's mass obtained from other sources.

It will be noticed from the expressions given above that the portions of the inequality, contained in the two parts of the disturbing function, have a strong tendency to cancel each other. This is always the case where either one of the mean anomalies is involved in the argument only to the simple multiple. This tendency does not occur in the inequalities having arguments of the form $7g - 2g'$, and perhaps quite large coefficients might be obtained for these in some of the asteroids whose periods approach $3\frac{1}{2}$ years, especially if their eccentricities are large. Melpomene would seem to afford the best chance, and the period of the inequality would have the recommendation of being much shorter than that of the one here computed, namely about 80 years.

MEMOIR No. 16.

Solution of a Problem in the Theory of Numbers.

(The Analyst, Vol. I, pp. 27-28, 1874.)

The following problem appeared in the *Mathematical Monthly*, Vol. I, p. 29, and no solution was published in that periodical:

"Show that the product of six entire consecutive numbers cannot be the square of a commensurable number."

Since the square root of every integer, not an exact square, is a surd, it will be sufficient to show that the product cannot be the square of an integer. Let the six numbers be denoted, n being an odd integer, by

$$\frac{n-5}{2}, \quad \frac{n-3}{2}, \quad \frac{n-1}{2}, \quad \frac{n+1}{2}, \quad \frac{n+3}{2}, \quad \frac{n+5}{2}.$$

Then it is required to prove the impossibility of $\frac{n^2-25}{4}\cdot\frac{n^2-9}{4}\cdot\frac{n^2-1}{4}=\square$.

Let us put $\frac{n^2-9}{4}=x$, where x is integral since it is the product of two integers. Then it will suffice to prove the impossibility of $x(x+2)(x-4)=\square$.

Let us suppose $x=k^2y$, where k^2 is the largest square factor contained in x, and thus y will be divisible by no square other than unity. Then we have to prove the impossibility of $y(k^2y+2)(k^2y-4)=\square$. But since y contains no square factor, both members of this equation must be divisible by y^2; this demands that $2\times 4=8$ be divisible by y. Hence, having regard to the restriction on the form of y, if the equation is possible, it can be so only for the values $y=1$, or $y=2$. The first gives $(k^2-1)^2-\square=9$, which is satisfied only by $k^2-1=5$, or $k=\sqrt{6}$, a surd; therefore y cannot be unity. For $y=2$ we have $2(k^2+1)(k^2+2)=\square$. But every square is of the form $3n$ or $3n+1$; if these are substituted in succession for k^2 in the left member of the last equation, it will be seen that the resulting quantities are of the form $3n+2$, and thus cannot be squares. Therefore y cannot be 2, and the impossibility is completely demonstrated.

Evidently the proposition might be enunciated in the much more general manner:

The product of any number of consecutive integers cannot be an exact power of any degree.

MEMOIR No. 17.

A Second Solution of the Problem of No. 8.

(The Analyst, Vol. I, pp. 43-46, 1874.)

Let Δ denote the distance of the planet from the earth. By the theory of the transformation of rectangular coordinates from the center of the sun as origin to the center of the earth, we shall have generally the two equations

$$\begin{aligned}\Delta \cos \lambda &= r \cos \chi + R \cos L, \\ \Delta \sin \lambda &= r \sin \chi + R \sin L,\end{aligned}$$

from which may be derived the two

$$\begin{aligned}\Delta \cos (\lambda - P) &= r \cos (\chi - P) + R \cos (L - P), \\ \Delta \sin (\lambda - P) &= r \sin (\chi - P) + R \sin (L - P),\end{aligned}$$

where P is any arbitrary angle. If we apply our equations to each of the three observations, we shall have the six equations,

$$\begin{aligned}\Delta_{-1} \cos \lambda_{-1} &= r \cos (\chi_0 - \eta) + R_{-1} \cos L_{-1}, \\ \Delta_{-1} \sin \lambda_{-1} &= r \sin (\chi_0 - \eta) + R_{-1} \sin L_{-1}, \\ \Delta_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0, \\ \Delta_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0, \\ \Delta_1 \cos \lambda_1 &= r \cos (\chi_0 + \eta) + R_1 \cos L_1, \\ \Delta_1 \sin \lambda_1 &= r \sin (\chi_0 + \eta) + R_1 \sin L_1.\end{aligned}$$

These equations contain the six unknowns Δ_{-1}, Δ_0, Δ_1, r, χ, η. If we eliminate Δ_{-1}, Δ_0, Δ_1 from them, we shall have the three equations of the first solution. But by retaining Δ_0 as the unknown, we shall arrive at an elegant solution. Let us first eliminate Δ_{-1} and Δ_1; this we do by putting $P = \lambda_{-1}$ for the first two equations, and $P = \lambda_1$ for the last two. The equations for determining the four remaining unknowns, are

$$\begin{aligned}0 &= r \sin (\chi_0 - \eta - \lambda_{-1}) + R_{-1} \sin (L_{-1} - \lambda_{-1}), \\ \Delta_0 \cos \lambda_0 &= r \cos \chi_0 + R_0 \cos L_0, \\ \Delta_0 \sin \lambda_0 &= r \sin \chi_0 + R_0 \sin L_0, \\ 0 &= r \sin (\chi_0 + \eta - \lambda_1) + R_1 \sin (L_1 - \lambda_1).\end{aligned}$$

If, in the second and third of these equations we put successively $P = \eta + \lambda_{-1}$ and $P = -\eta + \lambda_1$, we get

$$\Delta_0 \sin(\lambda_0 - \eta - \lambda_{-1}) = r \sin(\chi_0 - \eta - \lambda_{-1}) + R_0 \sin(L_0 - \eta - \lambda_{-1}),$$
$$\Delta_0 \sin(\lambda_0 + \eta - \lambda_1) = r \sin(\chi_0 + \eta - \lambda_1) + R_0 \sin(L_0 + \eta - \lambda_1).$$

If, from these equations we subtract the first and last of the preceding four, we get

$$\Delta_0 \sin(\lambda_0 - \eta - \lambda_{-1}) = R_0 \sin(L_0 - \eta - \lambda_{-1}) + R_{-1} \sin(L_{-1} - \lambda_{-1}),$$
$$\Delta_0 \sin(\lambda_0 + \eta - \lambda_1) = R_0 \sin(L_0 + \eta - \lambda_1) - R_1 \sin(L_1 - \lambda_1).$$

Two equations with two unknowns are thus arrived at without complicating the form of the original equations.

It is very easy to eliminate Δ_0 from these, and we get

$$[R_0 \sin(L_0 - \eta - \lambda_{-1}) - R_{-1} \sin(L_{-1} - \lambda_{-1})] \sin(\lambda_0 + \eta - \lambda_1)$$
$$= [R_0 \sin(L_0 + \eta - \lambda_1) - R_1 \sin(L_1 - \lambda_1)] \sin(\lambda_0 - \eta - \lambda_{-1}).$$

But we prefer to keep Δ_0 as our final unknown. Let us put for the sake of brevity

$$\eta = \sigma + \frac{\lambda_1 - \lambda_{-1}}{2}, \quad \delta = \lambda_0 - \frac{\lambda_1 + \lambda_{-1}}{2}, \quad \delta' = L_0 - \frac{\lambda_1 + \lambda_{-1}}{2},$$
$$\psi_{-1} = L_{-1} - \lambda_{-1}, \quad \psi_1 = L_1 - \lambda_1.$$

All these are known quantities with the exception of σ, which will take the place of η as an unknown. Our two equations can now be written

$$\Delta \sin(\delta - \sigma) = R_0 \sin(\delta' - \sigma) + R_{-1} \sin\psi_{-1},$$
$$\Delta_0 \sin(\delta + \sigma) = R_0 \sin(\delta' + \sigma) + R_1 \sin\psi_1.$$

Or, by taking in succession half the sum and half the difference

$$\Delta_0 \sin\delta \cos\sigma = R_0 \sin\delta' \cos\sigma + \tfrac{1}{2}(R_1 \sin\psi_1 + R_{-1} \sin\psi_{-1}),$$
$$\Delta_0 \cos\delta \sin\sigma = R_0 \cos\delta' \sin\sigma + \tfrac{1}{2}(R_1 \sin\psi_1 - R_{-1} \sin\psi_{-1}).$$

Whence

$$\cos\sigma = \tfrac{1}{2}\frac{R_1 \sin\psi_1 + R_{-1} \sin\psi_{-1}}{\Delta_0 \sin\delta - R_0 \sin\delta'},$$
$$\sin\sigma = \tfrac{1}{2}\frac{R_1 \sin\psi_1 - R_{-1} \sin\psi_{-1}}{\Delta_0 \cos\delta - R_0 \cos\delta'}.$$

By putting (these are all known quantities)

$$a = \frac{R_1 \sin\psi_1 + R_{-1} \sin\psi_{-1}}{2 \sin\delta}, \quad b = \frac{R_1 \sin\psi_1 - R_{-1} \sin\psi_{-1}}{2 \cos\delta}, \quad c = R_0 \frac{\sin\delta'}{\sin\delta}, \quad d = R_0 \frac{\cos\delta'}{\cos\delta},$$

we shall obtain the very elegant form for our final equation determining Δ_0,

$$\left\{\frac{a}{\Delta_0 - c}\right\}^2 + \left\{\frac{b}{\Delta_0 - d}\right\}^2 = 1.$$

This is, as we see, of the fourth degree in Δ_0; but in the case where the three right lines mentioned in the statement of the problem have a common point, this equation will have a root $\Delta_0 = 0$, that is, the absolute term of the equation will be 0; in this case, therefore, the equation reduces to the third degree.

By the introduction of the new unknown

$$x = \Delta_0 - \tfrac{1}{2}(c + d),$$

and putting $h = \frac{1}{2}(c - d)$, the equation takes the somewhat simpler form

$$\left(\frac{a}{x + h}\right)^2 + \left(\frac{b}{x - h}\right)^2 = 1.$$

or

$$(x^2 - h^2)^2 = a^2 (x - h)^2 + b^2 (x + h)^2.$$

MEMOIR No. 18.

Remarks on the Stability of Planetary Systems.

(The Analyst, Vol. I, pp. 53-60, 1874.)

As, in some quarters, quite erroneous views seem to be entertained regarding the conditions necessary for the stability of the solar system, it may be of service to note here, in brief, what is known on this subject.

It is remarkable that, although the meaning of stability in statics is well known, no one, so far as I know, has ever given a rigorous definition of this term as used in dynamics. As applied to the solar system, the sense attributed to it in general seems to involve the idea that the mean distances, eccentricities and mutual inclinations of the planets should always be comprised within narrow limits. But if this be the proper meaning of the word, one is tempted to ask—how narrow? It is plain, when we consider the matter more closely, that the distinction between stability and instability is one of kind and not of degree. There must be a sharp line separating stable systems from unstable.

In the first place we must discriminate between two possible significations of the term; a system may be stable or unstable with reference to the action of foreign forces, or with reference to the mutual action of its parts. A slight disturbance from without may cause in a moving system only trifling deviations from the previous paths of motion, or the effect may be a greater and still greater departure from them. This is quite analogous to the stability and instability of statics. But the stability of a planetary system, with reference to its own action, must be defined in a way quite peculiar.

A planetary system is stable when finite superior and inferior limits can be assigned to all the distances of the bodies composing it, and that, no matter how long the motion may be prolonged; but, if to some or all the distances, no superior limit other than infinity, or no inferior limit other than zero can be assigned, the system is unstable.

Hence, in a stable system, there can be no collision and no indefinite separation of the bodies composing it. With this definition of stability we see that, in the problem of two bodies, motion in an ellipse is stable, but motion in a right line, or parabola, or hyperbola, is unstable.

With regard to stable systems, we may enunciate the following proposition: The coordinates of all the bodies in a stable system, or any function of them, which remain always finite and continuous, can be developed in infinite converging series of periodic terms, each of the form $K \frac{\sin}{\cos}(kt+\beta)$: K, k and β being absolutely constant; and the argument $kt + \beta$ is always composed, as a linear function with positive or negative integral coefficients, of other arguments, whose number never exceeds $3n - 3$, n being the number of bodies in the system.*

With regard to the convergence of these series, it must be understood that it is asserted only when K and k are taken as wholes; if these quantities are expressed in infinite series involving the powers and products of certain parameters, these series may cease to be convergent long before the system passes from stability to instability. But the convergence of these is altogether another question.

If mathematicians had succeeded in completely integrating in finite terms the differential equations of motion of a system of material points acting on each other in accordance with the law of gravitation, the conditions under which the motion is stable or unstable could be immediately assigned. But there is scarcely any reason to expect that this will ever be accomplished, and perhaps it is in the power of analysis, at present even, to demonstrate its impossibility. For, just as, from the fact that the equation, $\sin x = 0$, has an infinite number of roots, it may be confidently asserted that $\sin x$ cannot be represented in finite terms by algebraic functions; or that, because an elliptic function possesses the property of double periodicity, it cannot be equivalent to any finite expression involving circular or logarithmic functions; so it is probable that the functions, defined by the differential equations of planetary motion, have properties that cannot belong to any finite expression involving quadratures. If this should be the case, all attempts to arrive at a complete solution, in finite terms, of the famous problem of three bodies, must prove as abortive as those made to square the circle, or to express elliptic integrals in circular and logarithmic functions.

The solution of the general problem, given the initial positions and velocities of a system of material points, to determine whether the ensuing motion is stable or unstable, in the sense we have attributed to these words, does not seem to have engaged the attention of geometers. It does not, however, demand the complete integration of the differential equations of

* The researches of M. Poincaré have since shown the inexactitude of this.

motion. Thus, where the system contains two material points only, two integrals are needed, that of the conservation of living forces, and that of the conservation of areas; it depends on the values of the constants annexed to complete these integrals; to insure stability the first must be negative and the second must not vanish.

The question of stability is intimately connected with the values of the coefficients k in the periodic terms of the series into which the coordinates can be developed in the case of stable motion. If we substitute for the coordinates in the differential equations indeterminate series of this form, we shall arrive at a number of equations exactly sufficient to determine all the coefficients K and the $3n - 3$ independent k's in terms of $3n - 3$ arbitrary constants which must be given by the initial state of the system. Eliminating all the coefficients K, we shall have left $3n - 3$ equations determining the same number of independent k's. It can be readily shown that in these equations k appears only in even powers. Then, if the initial positions and velocities are such that they make these equations afford positive and finite values for all the quantities k^2, and if it be granted that these are the proper roots to take, the motion of the system is undoubtedly stable. But if these equations afford negative values for some or all of these quantities, and it be granted that these are the proper roots to take, the system is necessarily unstable. In the case where some of the quantities k^2 vanish, the question of stability must be determined from other considerations. Whether in any case it is proper to take the imaginary roots of these equations for the quantities k^2, or whether, like some analogous equations in the theory of heat, they have no roots of this kind, is a point which is not yet clear. In all this, as the equations determining the k's can be obtained only in the form of infinite series, it must be shown beforehand that the constants, defining the initial positions and velocities of the bodies of the system, enter into these equations in such a way that they render the series convergent; else any conclusions, as to the values of k's, deduced from them are not legitimately established.

The number of conditions, necessary to insure stability, of course increases with the number of bodies composing the system. In all cases, the constant annexed to the integral of living forces in relative motion must be negative. It is needless to say that in the solar system this condition is fulfilled. Certain popular writers have got it that incommensurability of mean motions is a *sine qua non* of stability; but I am not aware that this has been asserted by any geometer or astronomer of note.

This mistake doubtless arose from noticing that the near approach of mean motions to commensurability produces inequalities having large

coefficients through the division by the small mean motion of the argument. But it will not do to assume that these coefficients increase beyond every limit when the mean motion of the argument diminishes without limit, or that when this vanishes, there are terms in the planetary elements proportional to the time.

Let us illustrate this point more at length. If a is the mean distance of one of the planets, and θ an argument whose mean motion nearly or altogether vanishes; then, so far as this argument is concerned, we may have the equation

$$\frac{da}{dt} = A \sin\theta .$$

If the mean motion of θ is supposed to vanish, the integral of this equation is often written $a = a_0 + A \sin\theta . t$, a_0 being the value of a at the origin of time. But, although this treatment is allowable when we wish to find the value of a for small values of t, it will not answer when the object is to discover whether a is a periodic function of t or not. For it has been assumed that A is constant, whereas it is a function, not only of a, but of all the other varying elements which define the dimensions of the orbits; also θ is not constant, although its mean motion vanishes, for its periodic inequalities may have some effect here. If the approximation were carried further, it would be found that there were terms in a multiplied by t^2, t^3, etc., and that thus it would be more exact to write

$$a = a_0 + A \sin\theta . t + Bt^2 + Ct^3 + \ldots$$

What if the right member of this equation should turn out to be the development of a periodic function of t? This, in fact, is the result in a large class of cases. Thus it is plain that the equation

$$\frac{da}{dt} = A \sin\theta ,$$

must be treated as a differential equation; that is, its right member must be regarded as an unknown function of t as well as its left.

But several instances in the solar system of commensurability of mean motions, without resultant instability, ought to have prevented this mistake. The three inner satellites of Jupiter have mean motions generally granted to be exactly commensurable, yet the system is not supposed to be unstable. Again, what prevents our regarding the moon as a planet revolving around the sun, and our attributing its being sometimes in advance, sometimes behind the earth, its having a radius vector, sometimes greater, sometimes

less than that of the earth, to the perturbing influence of the latter? In this view the period of revolution of the moon about the sun is precisely equal to that of the earth; yet there is no instability here. If the planet Venus were moved outward from the sun, until its mean distance from this body became nearly equal to that of the earth, and, if at the same time their eccentricities and longitudes of perihelia were so nearly equal as to permit their being for some time in the vicinity of each other, the effect of their mutual action would be to make the mean values of these elements rigorously equal, and each planet would become a satellite to the other. Instability would not result from this disposition.

There are, moreover, two remarkable particular solutions of the problem of three bodies, in both of which the periods of revolution of the two planets are exactly equal, without instability ensuing. These solutions have been developed by Laplace (*Mécanique Céleste*, Book X, Chap. VI).

The first, in its stable form, may be stated thus: Two planets may move in the same direction about the sun in two equal ellipses, lying in the same plane, having their foci at the center of the sun, and their greater axes inclined at an angle of 60°, provided they are at the same time in corresponding points of their orbits, so that they, together with the sun, are always at the vertices of an equilateral triangle. The laws of motion are the same as in the case of two bodies, but the common mean motion is given by formula,

$$n = \sqrt{\frac{m + m' + m''}{a^3}},$$

where m, m', m'' are the three masses, and a the common mean distance.

The second is stated thus: Two planets may revolve in the same direction about the sun, in similar ellipses, having their foci at the center of this body, and their greater axes coincident in direction, provided that the ratio of the axes is determined by a root of a certain equation of the fifth degree involving in its coefficients the ratios of the three masses. The planets must be at corresponding points in their orbits at the same time, so that they and the sun always lie in a right line. The laws of motion are the same as for two bodies, but the common mean motion is given by a complex expression involving the root of the equation just mentioned.

It may be noticed that Liouville has shown that the latter of these solutions is unstable in the sense we first attributed to this word, that is with reference to slight disturbances from without.* It is probable that the

* See Connaissance des Temps for 1845, or Liouville's Journal, First Series, Vol. VII, p. 110.

first solution is in the same case, but I do not know that it has ever been discussed in this respect.*

Let us now establish in a clearer light the fact that commensurability of mean motions does not necessarily produce instability. The solution of the problem of three bodies can be reduced to the integration of a system of eight differential equations of the first order; and by a suitable selection of variables, these may be made to take the canonical shape; that is, the differential coefficient of each varying element, with respect to the time, will be equal to the positive or the negative of the partial differential coefficient, with respect to the conjugate element, of a function R, analogous to, but not identical with the disturbing function in perturbations. Thus the eight elements are divided into two classes; four, being functions of the mean distances and eccentricities, relate to the dimensions of the two orbits; while the other four, their conjugates, are simply the elementary arguments of the periodic terms contained by R in its developed form. The selection of these last may be made arbitrarily. If we take one of them, as θ, to coincide with an argument of R, whose mean motion nearly or exactly vanishes, and call the element conjugate to this, Θ, we shall have the two differential equations

$$\frac{d\Theta}{dt}=\frac{\partial R}{\partial\theta},\quad \frac{d\theta}{dt}=-\frac{\partial R}{\partial\Theta}.$$

Let us now suppose that R is reduced to its terms which have only θ in their arguments; then

$$R=-B-A\cos\theta-A'\cos 2\theta-A''\cos 3\theta-\ldots,$$

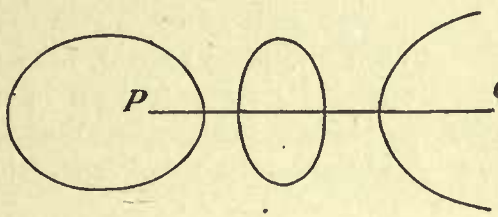
where B, A, etc., are functions of Θ and the three other elements of its class. As R thus limited does not contain the three elements which are in the class of θ, its partial differential coefficients, with respect to these quantities, vanish. Then the three elements accompanying Θ in its class are constant, and R, as we have limited it, contains no other variables than Θ and θ. Thus, if the differential equations determining Θ and θ are multiplied, the first by $d\theta$, and the second by $-d\Theta$, and the results added, we have an exact differential, which being integrated, gives $R=$ a constant, or, as it may be written,

$$C=B+A\cos\theta+A'\cos 2\theta+A''\cos 3\theta+\ldots$$

In order to obtain the values of Θ and θ in terms of t, we should have to make another integration, but this integral suffices to show whether the

* Since this was written, the discussion has been made, and stability depends on whether the masses of the three bodies fulfil a certain condition.

element Θ, on which depend the dimensions of the orbits, is confined between finite limits. The value of the constant C is readily obtained by substituting in the right member of the equation the values of θ, Θ and the three other elements of its class, which have place at a determinate time, as, for instance, the epoch from which t is counted. Then this equation may be regarded as the polar equation of a curve, upon which the values of Θ and θ are always found together. Let us suppose that, Θ being taken as the radius and θ as the angle, the equation is represented by a curve having one or more of such branches as those in the figure. P is the pole from which Θ is measured, and PQ the line from which θ is measured. Now if the values of Θ and θ, at a determinate time, are found on the closed branch which envelops the pole P, it is plain Θ will always be comprised between certain finite limits. And, in this case, the mean motion of the argument cannot vanish, as θ moves through the entire circumference. Here it is always possible to develop Θ and θ in converging infinite series consisting of periodic terms, such as



$$\Theta = \Theta_0 + \Theta_1 \cos [\theta_0 (t + c)] + \Theta_2 \cos 2 [\theta_0 (t + c)] + \ldots,$$
$$\theta = \theta_0 (t + c) + \theta_1 \sin [\theta_0 (t + c)] + \theta_2 \sin 2 [\theta_0 (t + c)] + \ldots,$$

where Θ_0, Θ_1, Θ_2, ..., θ_0, θ_1, θ_2, ... and c are constants. These are the series of which Delaunay has made such constant use in his Theory of the Motion of the Moon.

But if the values of Θ and θ, at a determinate time, are found upon the closed branch holding the middle place in the figure, Θ will always be contained within finite limits, while θ, its mean motion vanishing, will make oscillations forth and back between definite limits. Hence, although the mean motions are here exactly commensurable, no instability results. This case obtains in the three inner satellites of Jupiter, and it also has place in the system of the sun, earth and moon, when the last, as well as the earth, is regarded as a planet circulating about the sun.

[In the third place, the curve, upon which are found the values of Θ and θ, may have infinite branches, such as those of the curve at the right side of the figure. Here Θ, coming from infinity, would tend to the same, and thus, dependent however on the signification of Θ, instability may be indicated.]*

* This paragraph was inadvertently omitted in the original memoir.

In all this we must remember that the values of Θ and the three other elements of its class, both at the origin of time and ever before and after, must be such that they allow the development of R in periodic series to be convergent; else any conclusions derived from these series are not legitimately established.

From this we see that commensurability or incommensurability of mean motions has no marked connection with stability. This last may be said to depend rather on whether the elements, such as the mean distances and eccentricities, which determine the dimensions of the orbits, have, at a given time, such values as make the latter depart but little from the circular form, and permit to them the vibrations caused by the action of the members of the system, without interference, or, in other words, intersections. There can hardly be a doubt that our solar system as composed of the sun and eight principal planets, fulfils these conditions.

MEMOIR No. 19.

Useful Formulas in the Calculus of Finite Differences.

(The Analyst, Vol. I, pp. 141-145, 1874; Vol. II, pp. 8-9, 1875.)

The finding of the values of the differential coefficients of a function of a single variable, and of the single and double integrals with respect to the independent variable, from special values of the functions computed at equidistant intervals, is an operation very frequent in planetary astronomy. The following seems a simpler exposition of the matter than has hitherto been given:

Let y be a function of x computed for the series of values of $x, \ldots . a-h$, $a, a+h, a+2h, \ldots .$; and let the differences and first and second summed values of y be denoted thus:

. . . .

$a-h$	$\Delta^{-2}y_{-1}$	$\Delta^{-1}y_{-\frac{3}{2}}$	y_{-1}	$\Delta y_{-\frac{3}{2}}$	$\Delta^2 y_{-1}$	$\Delta^3 y_{-\frac{3}{2}} \ldots$
a	$\Delta^{-2}y_0$	$\Delta^{-1}y_{-\frac{1}{2}}$	y_0	Δy_{-}	$\Delta^2 y_0$	$\Delta^3 y_{-\frac{1}{2}} \ldots$
$a+h$	$\Delta^{-2}y_1$	$\Delta^{-1}y_{\frac{1}{2}}$	y_1	$\Delta y_{\frac{1}{2}}$	$\Delta^2 y_1$	$\Delta^3 y_{\frac{1}{2}} \ldots$
$a+2h$	$\Delta^{-2}y_2$	$\Delta^{-1}y_{\frac{3}{2}}$	y_2	$\Delta y_{\frac{3}{2}}$	$\Delta_2 y_2$	$\Delta^3 y_{\frac{3}{2}} \ldots$

. . . .

With regard to the differences of odd orders, let us adopt the general notation,

$$\Delta^{2n+1}y_i = \tfrac{1}{2}(\Delta^{2n+1}y_{i-\frac{1}{2}} + \Delta^{2n+1}y_{i+\frac{1}{2}}),$$

n and i being integers. In this way the symbol Δ does not follow the law of indices as in the ordinary method of differences; that is, we do not have in general $\Delta^n \Delta^{n'} = \Delta^{n+n'}$. Nevertheless it is evident the following relations hold:

$$\Delta^{2n}\Delta^{2n'} = \Delta^{2(n+n')}, \qquad \Delta^{2n+1}\Delta^{2n'} = \Delta^{2(n+n')+1},$$

that is, the exponents are to be added except when both are odd.

For brevity, writing D for $\frac{d}{dx}$, and e denoting the base of hyperbolic logarithms, the symbolical expression for Taylor's Theorem gives

$$y_{-1} = e^{-hD}y_0, \quad y_0 = y_0, \quad y_1 = e^{hD}y_0.$$

Whence it is easy to see that

$$\Delta = \tfrac{1}{2}[e^{hD} - e^{-hD}], \quad \Delta^2 = e^{hD} + e^{-hD} - 2.$$

The last may be written

$$\Delta^2 = (e^{\frac{hD}{2}} - e^{-\frac{hD}{2}})^2.$$

Thus it is evident that we have, in general,

$$\Delta^{2n} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n}, \quad \Delta^{2n+1} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n+1} \frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2}.$$

Also,

$$e^{\frac{hD}{2}} - e^{-\frac{hD}{2}} = \sqrt{\Delta^2}, \quad e^{\frac{hD}{2}} + e^{-\frac{hD}{2}} = \sqrt{4 + \Delta^2}.$$

Whence

$$h^2D^2 = 4 \log^2\left(\frac{\Delta}{2} + \sqrt{1 + \tfrac{1}{4}\Delta^2}\right) = \left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^2,$$

the integral being taken so as to vanish with Δ. The operation denoted by the symbolical expression $\frac{\Delta}{D}$ is evidently a function of Δ^2, and we have

$$\frac{\Delta}{hD} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)\frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2hD} = \frac{\Delta\sqrt{1 + \frac{1}{4}\Delta^2}}{hD} = \Delta\sqrt{1 + \tfrac{1}{4}\Delta^2}\left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^{-1}.$$

Whence

$$hD = \frac{1}{\sqrt{1 + \frac{1}{4}\Delta^2}} \int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}.$$

It is plain, then, that we have, in general, the value of an even differential coefficient from the formula

$$D^{2n} = h^{-2n}\left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^{2n},$$

and the value of an odd one from

$$D^{2n+1} = \frac{h^{-2n-1}}{\sqrt{1 + \frac{1}{4}\Delta^2}}\left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^{2n+1}.$$

Differentiating the first of these with respect to Δ, we obtain

$$\frac{d \,.\, D^{2n}}{d\Delta} = \frac{2nh^{-2n}}{\sqrt{1 + \frac{1}{4}\Delta^2}}\left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^{2n-1} = 2nh^{-1}D^{2n-1}.$$

Thus the value of an even differential coefficient can be obtained from that of the preceding differential coefficient by the very simple formula

$$D^{2n} = 2nh^{-1}\int D^{2n-1}d\Delta.$$

If we employed the preceding formula for D for expanding its value in powers of Δ, we should find it difficult to discover the law of the numerical coefficients, but by differentiating the value of D^{2n+1} with respect to Δ, we shall find that it satisfies the differential equation

$$[1 + \tfrac{1}{4}\Delta^2]\frac{d \,.\, D^{2n+1}}{d\Delta} + \frac{\Delta}{4} D^{2n+1} = (2n+1)\, h^{-1} D^{2n},$$

which, when $n = 0$, becomes

$$[1 + \tfrac{1}{4}\Delta^2]\frac{dD}{d\Delta} + \frac{\Delta}{4} D = h^{-1}.$$

If, in this, we suppose

$$D = h^{-1}\Sigma \,.\, A_n \Delta^n,$$

we shall find that the coefficients satisfy in general the relation

$$(n+2)\, A_{n+2} + \frac{n+1}{4} A_n = 0,$$

whence

$$A_{n+2} = -\frac{n+1}{4(n+2)} A_n.$$

As we know that $A_1 = 1$, this suffices for obtaining all the coefficients in succession. In the general case, if we put

$$D^{2n} = h^{-2n}\Sigma \,.\, A_i^{(2n)}\Delta^i, \qquad D^{2n+1} = h^{-2n-1}\Sigma \,.\, A_i^{(2n+1)}\Delta^i,$$

the differential equation above gives the following relation between the coefficients:

$$A_{i+2}^{(2n+1)} = -\frac{i+1}{4(i+2)} A_i^{(2n+1)} + \frac{2n+1}{i+2} A_{i+1}^{(2n)},$$

by which all the coefficients in succession may be derived.

We have

$$D = h^{-1}\left(\Delta - \frac{1}{3}\frac{\Delta^3}{2} + \frac{1.2}{3.5}\frac{\Delta^5}{2^2} - \frac{1.2.3}{3.5.7}\frac{\Delta^7}{2^3} + \ldots\right),$$

$$D^2 = h^{-2}\left(\Delta^2 - \frac{1}{3.2}\frac{\Delta^4}{2} + \frac{1.2}{3.5.3}\frac{\Delta^6}{2^2} - \frac{1.2.3}{3.5.7.4}\frac{\Delta^8}{2^3} + \ldots\right),$$

where the law of the coefficients is readily seen. In the higher differential coefficients, the fractions are more complex; we therefore content ourselves with writing the values thus:

$$D^3 = h^{-3}(\Delta^3 - \tfrac{1}{4}\Delta^5 + \tfrac{7}{120}\Delta^7 - \tfrac{41}{1024}\Delta^9 + \ldots),$$
$$D^4 = h^{-4}(\Delta^4 - \tfrac{1}{6}\Delta^6 + \tfrac{7}{240}\Delta^8 - \tfrac{41}{7560}\Delta^{10} + \ldots),$$
$$D^5 = h^{-6}(\Delta^5 - \tfrac{1}{3}\Delta^7 + \tfrac{13}{144}\Delta^9 - \ldots),$$
$$D^6 = h^{-8}(\Delta^8 - \tfrac{1}{4}\Delta^8 + \tfrac{13}{240}\Delta^{10} - \ldots).$$

The expressions given for D^{2n} and D^{2n+1} are equally applicable when n is negative; they then give the formulas to be used in mechanical quadratures, thus:

$$D^{-1} = \frac{h}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-1},$$

$$D^{-2} = h^2\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-2}.$$

If these expressions are expanded in powers of Δ, we obtain

$$D^{-1} = h\left(\Delta^{-1} - \frac{1}{12}\Delta + \frac{11}{720}\Delta^3 - \frac{191}{60480}\Delta^5 + \frac{2497}{3628800}\Delta^7 - \frac{14797}{95800320}\Delta^9 + \frac{92427157}{2615348736000}\Delta^{11} - \ldots\right),$$

$$D^{-2} = h^2\left(\Delta^{-2} + \frac{1}{12}\Delta^0 - \frac{1}{240}\Delta^2 + \frac{31}{60480}\Delta^4 - \frac{289}{3628800}\Delta^6 + \frac{317}{22809600}\Delta^8 - \ldots\right).$$

These are the expressions to be used in computing the values of the integrals $\int y dx$ and $\int\int y dx^2$. It must be noticed that Δ^{-1} virtually contains an arbitrary constant C, and Δ^{-2} an arbitrary expression $Cx + C'$. In fact, the quantities in the columns to the left of that of the function y cannot be written until we know one quantity in each column. These constants C and C' are usually determined from the given values of $\int y dx$ and $\int\int y dx^2$ for $x = a$. If we denote them by D_0^{-1} and D_0^{-2}, and if, in general, the subscript ($_0$) denote values which obtain when $x = a$, it will be seen that

$$\Delta_0^{-1} = \frac{D_0^{-1}}{h} + \frac{1}{12}\Delta_0 - \frac{11}{720}\Delta_0^3 + \ldots,$$

$$\Delta_0^{-2} = \frac{D_0^{-2}}{h^2} - \frac{1}{12}\Delta_0^0 + \frac{1}{240}\Delta_0^2 - \ldots.$$

Having thus the sum and difference of the quantities $\Delta^{-1}y_{-\frac{1}{2}}$ and $\Delta^{-1}y_{\frac{1}{2}}$, it will be easy to get the quantities themselves.

The preceding formulas give the values of the integrals for the series of values of x, .. $a - h$, a, $a + h$, .. It is generally preferable to compute them for the values, ... $a - \frac{1}{2}h$, $a + \frac{1}{2}h$, $a + \frac{3}{2}h$, ... Formulas for this purpose can be obtained by the simple consideration, that in the scheme, given at the beginning of this article, it is allowable to treat the odd orders of differences as if they were even, and the even as if they were odd.

In this way all the quantities obtained will correspond to the middle of the intervals of the former supposition. Thus, calling D^{-1} and D^{-2} in this case $D_{\frac{1}{2}}^{-1}$ and $D_{\frac{1}{2}}^{-2}$, it is evident we must have

$$D_{\frac{1}{2}}^{-1} = h\left(\int \frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-1},$$

$$D_{\frac{1}{2}}^{-2} = \frac{h^2}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int \frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-2},$$

or, expanded in powers of Δ,

$$D_{\frac{1}{2}}^{-1} = h\left(\Delta^{-1} + \frac{1}{24}\Delta - \frac{17}{5760}\Delta^3 + \frac{367}{967680}\Delta^5 - \frac{27859}{464486400}\Delta^7 + \ldots\right),$$

$$D_{\frac{1}{2}}^{-2} = h^2\left(\Delta^{-2} - \frac{1}{24}\Delta^0 + \frac{17}{1920}\Delta^2 - \frac{367}{193536}\Delta^4 + \frac{27859}{66355200}\Delta^6 - \ldots\right).$$

The differences of the first formula, although they are of odd orders, are to be taken as equivalent to the simple numbers standing in the original scheme, while the differences of the second, although of even orders, are all the averages of two adjacent numbers of the same scheme.

It is plain we have

$$D_{\frac{1}{2}}^{-2} = -h\frac{d\,.\,D_{\frac{1}{2}}^{-1}}{d\Delta}.$$

In using the method of mechanical quadratures, it is usual to multiply the values of y by h, if the single integral only is wanted, but by h^2 if the double is also to be obtained; in the last case then it is necessary to divide the results obtained by h in order to have the single integral.

These formulas appear to have been first obtained by Gauss (*Werke*, Vol. III, p. 328). Encke has given them in the Berlin *Jahrbuch* for 1838. For use they are much superior to the formula given by Laplace (*Mécanique Céleste*, Vol. IV, p. 207).

MEMOIR No. 20.

Elementary Treatment of the Problem of Two Bodies.

(The Analyst, Vol. I, pp. 165-170, 1874.)

The deduction of the motion of the planets, in accordance with the laws of Kepler, from the principle of universal gravitation, is important, not only on account of the extensive rôle this theory plays in Astronomy, but also for its interest, in a historical point of view, as Newton's principal discovery. Hence it is desirable that the demonstration should be made as elementary and as brief as possible, in order that it may be brought within the comprehension of the largest number of persons.

The polar equation of the conic section, referred to a focus as pole

$$r = \frac{a(1-e^2)}{1 + e\cos(\lambda - \omega)},$$

is well known ; a denotes half the greater axis, e the eccentricity and ω the angle made by the axis with the line from which λ is measured. It will be advantageous to replace $a(1-e^2)$ by p, p being the semi-parameter, also to put

$$\alpha = e\cos\omega, \quad \beta = e\sin\omega.$$

Thus the equation becomes

$$r + \alpha r\cos\lambda + \beta r\sin\lambda = p.$$

Hence it is plain that the equation, in terms of rectangular coordinates, the origin being at a focus, but the axes of coordinates having any direction we please, is

$$\sqrt{x^2 + y^2} + \alpha x + \beta y = p. \tag{1}$$

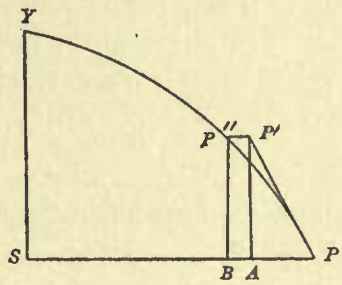
We take for granted the following theorems, since they are demonstrated in the most elementary treatises on mechanics:

In determining the relative motion of one body about another, it suffices to regard the latter as fixed, and to attribute to it a mass equal to the sum of the masses, and then to suppose the moving body without mass.

When a body describes a plane curve, and the radius vector, drawn from a fixed point in the plane of the curve, passes over equal areas in

equal times (which we shall express by saying that the areolar velocity about the fixed point is constant), the force acts always in the direction of the radius; and the converse.

Now let a body describe a conic section about another occupying a focus, the areolar velocity about this focus being constant; it is required to determine the force acting.



In the figure, let $PP''Y$ be an arc of the conic section so described, S being the focus. Let P and P'' be any two points on the curve at an indeterminate but small distance from each other. Draw SP, and PP' a tangent at P, $P''P'$ parallel to, and $P'A$ and $P''B$ perpendicular to SP. Let SP be taken as the axis of x, and SY perpendicular to it, as the axis of y. The coordinates of P are then $x = SP = r_0$, $y = 0$; substituting these in the equation of the curve, we get

$$(1+\alpha)\, r_0 = p. \tag{2}$$

Since the ordinate y can here be supposed always very small, the term $\sqrt{x^2+y^2}$ in (1) can be expanded, by the binomial theorem, in a series of ascending powers of y. Neglecting y^4 and higher powers, we get

$$x + \tfrac{1}{2}\frac{y^2}{x} + \alpha x + \beta y = p,$$

or, as x differs from r_0 only by a quantity of the order of y, by neglecting y^3

$$x + \tfrac{1}{2}\frac{y^2}{r_0} + \alpha x + \beta y = p, \quad x = \frac{p - \beta y - \tfrac{1}{2}\frac{y^2}{r_0}}{1+\alpha}.$$

Or, by (2),

$$x = r_0 - \frac{\beta}{1+\alpha} y - \tfrac{1}{2}\frac{y^2}{p}.$$

This is the value of x from (1) expanded in a series of ascending powers of y, the cube and higher powers being omitted. The equation

$$x = r_0 - \frac{\beta}{1+\alpha} y$$

belongs to a right line, which can be nothing else than the tangent PP'. Hence it is plain, from the figure, that taking $P''B = P'A = y$,

$$\tan PP'A = \frac{\beta}{1+\alpha}, \tag{3}$$

$$PA = \frac{\beta}{1+\alpha} y,$$

$$P'P'' = AB = \tfrac{1}{2}\frac{y^2}{p}, \tag{4}$$

the last equation being only approximate, but more and more nearly true as $P''B$ or y becomes smaller.

Let F denote the force acting on the moving body, and t the small interval of time in which the latter passes from P to P''. Then we have

$$P'P'' = \tfrac{1}{2}\frac{y^2}{p} = \tfrac{1}{2}Ft^2.$$

If we denote double the areolar velocity by h, since $P''B = y$ is very small, we have

$$SP \,.\, P''B = r_0 y = ht.$$

Eliminating t from these equations, we get

$$F = \frac{h^2}{pr^2}.$$

Since there is no limit to the supposed smallness of y and t, this equation is rigorously exact. The force is then inversely as the square of the radius-vector, and its intensity at the unit of distance is found simply by dividing the square of double the areolar velocity by the semi-parameter. It is evidently attractive except when, the motion being in a hyperbola, the focus, about which the areolar velocity is constant, is the exterior, in which case it is repulsive.

Taking up the inverse problem, let a body start from P towards P' with a velocity v, which would carry it to the latter point in the time t, and let it be subjected to the action of a force varying inversely as the square of its distance from a second body supposed fixed at S: it is required to find the curve described.

Let the masses of the bodies, measured by the velocities they are able to communicate by their action, in the unit of time and at the unit of distance, be denoted severally by m and M. The force acting at P is then

$$\frac{M+m}{\overline{SP}^2} = \frac{M+m}{r^2},$$

and, if at the end of the time t, the body is at P'' instead of P', we must have

$$P'P'' = \tfrac{1}{2}\frac{M+m}{r^2}t^2.$$

But, as before, the constancy of the areolar velocity gives $rx = ht$. Whence

$$P'P'' = \frac{M+m}{2h^2} y^2.$$

This equation coincides with (4) if we suppose

$$p = \frac{h^2}{M+m}. \tag{5}$$

Let now a conic section, having this value for its semi-parameter, be described with S as focus and touching PP' at P. That this is possible is evident from the general equation (1); here are only two unknowns, α and β, to be determined, and they are given by equations (2) and (3), whence we see the solution is always unique. A body, moving upon this conic section, would have, at the point P, the same velocity, and the same direction of motion, and be subjected to the action of an equal force having the same law of variation, as the moving body in the problem. Hence, if the path of the latter is thoroughly determinate, and it would be absurd to suppose otherwise, the conic section just described must be the curve sought.

We can easily find the elements of this conic section. Thus, let the angle $P'PS$ be denoted by ψ, then evidently,

$$h = rv \sin\psi,$$

which, substituted in (5), gives the value of $p = a(1 - e^2)$; next α and β, which we recall stand for $e \cos\omega$ and $e \sin\omega$, are given by (2) and (3). That is,

$$a(1-e^2) = \frac{r^2v^2 \sin^2\psi}{M+m},$$

$$e\cos\omega = \frac{rv^2 \sin^2\psi}{M+m} - 1,$$

$$e\sin\omega = \frac{rv^2 \sin\psi \cos\psi}{M+m},$$

whence we derive

$$e^2 = 1 - 2\frac{rv^2 \sin^2\psi}{M+m} + \frac{r^2v^4 \sin^2\psi}{(M+m)^2}, \quad \frac{1}{a} = \frac{2}{r} - \frac{v^2}{M+m}.$$

Consequently the greater axis, and the species of conic section described, are independent of ψ. We have an ellipse, a parabola, or a hyperbola, according as v^2 is less, equal to, or greater than $2\,\dfrac{M+m}{r}$.

From the last equation

$$v^2 = (M+m)\left(\frac{2}{r} - \frac{1}{a}\right), \tag{6}$$

which may evidently be taken as a general expression for the square of the velocity, if r denote the general radius vector.

Also from (5),

$$h = \sqrt{(M+m)p}.$$

Thus, in different orbits, the areolar velocities are as the square roots of the parameters, and as the square roots of the sums of the masses. In an elliptic orbit, if T denote the time of revolution, the double of the area of the whole ellipse

$$hT = 2\pi a^2 \sqrt{1-e^2} = 2\pi a^{\frac{3}{2}} p.$$

Whence

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}.$$

Thus the theorem that, provided the sum of the masses remains the same, the squares of the periods in different orbits are as the cubes of the greater axes.

The mean angular velocity is usually denoted by n; thus

$$n = \frac{2\pi}{T} = \sqrt{\frac{M+m}{a^3}}.$$

It is customary with astronomers to assume the earth's mean distance from the sun as the linear unit. If M and m are the masses severally of the sun and earth, and m', a' and n' belonging to another planet are introduced, the mean distance of the last is given by the equation

$$a'^3 = \frac{1+\frac{m'}{M}}{1+\frac{m}{M}} \frac{n^2}{n'^2}.$$

To complete the subject, it is necessary to notice a particular case of the problem, viz., when $\psi = 0$. Here the motion is in a right line, and from (6) it appears the velocity is infinite when the body arrives at S. As the existence of another body here ought not to be considered, at least in a mathematical sense, as an obstacle to its further motion, it is plain the body will pass beyond and move in the same right line until its velocity is reduced to zero, when it will return on its path, which will thus be a portion of a right line of which S is the middle point. This cannot be considered as a degenerate form of a conic section of which S is the focus. For when an ellipse is varied by augmenting the eccentricity but maintaining the greater axis constant, at the point the first has attained the limit unity, the ellipse has degenerated into two equal portions of right lines overlapping

each other and having their extremities on one side in the point S. Hence this case must be regarded as a singular solution. However, most of the properties of motion can be deduced from those of elliptic motion. Thus, if the length of the whole path denoted by $4a$, the duration of an oscillation will be

$$\frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}.$$

Whence we gather that the time, in which a planet, at rest at its mean distance, would fall to the sun, is found by dividing its periodic time by $4\sqrt{2}$.

MEMOIR No. 21.

The Differential Equations of Dynamics.

(The Analyst, Vol. I, pp. 200-203, 1874.)

The general formula of dynamics is

$$\Sigma\left[\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right]=0.$$

In the usual treatment of this equation, we have been asked to attribute to the symbols δx, δy, δz, ... the signification they have in the calculus of variations. This, however, is unnecessary, except when we wish to deduce from it the principle of least action; and the student unacquainted with this calculus may regard these symbols as multipliers, which, when all the points of the system are free, have any finite values we please, but when the coordinates are restricted to satisfy an equation $U=0$, are subject to the condition

$$\frac{\partial U}{\partial x}\delta x+\frac{\partial U}{\partial y}\delta y+\frac{\partial U}{\partial z}\delta z+\ldots=0,$$

an equation which, for brevity, we shall write $\delta U=0$.

We shall confine our attention to those cases in which the equations of condition and the accelerating forces are functions of the coordinates and the time only, and in which the latter are equivalent to the partial differential coefficients of a single function Ω taken with respect to the coordinates whose acceleration they express.

Whenever a function as U involves, in addition to x, y, z, ... their differential coefficients with respect to the time, quantities which we shall denote by x', y', z', ..., we shall suppose that δU involves, besides the terms written above, the following

$$\frac{\partial U}{\partial x'}\delta x'+\frac{\partial U}{\partial y'}\delta y'+\frac{\partial U}{\partial z'}\delta z'+\ldots$$

Moreover, as we shall have to differentiate such functions as δU with respect to t, we shall meet with such quantities as $\frac{d\delta x}{dt}$, and shall suppose that the order of the symbols d and δ may be inverted, that is, we shall have equations such as

$$\frac{d\delta x}{dt}=\delta\frac{dx}{dt}=\delta x'.$$

The reader will see in this only a notational assumption, without quantitative significance, serving merely as machinery of demonstration. It will be noted that t is a variable not subject to the operation δ.

We have

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta\Omega,$$

and for convenience may put

$$\tfrac{1}{2}\Sigma m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right) = T.$$

Then it will readily be perceived that the general formula can be written thus

$$\frac{d}{dt}\cdot\Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right) - \delta(T+\Omega) = 0.$$

The coordinates $x, y, z, \ldots$, can be expressed as functions of the time and certain variables q_i, independent of each other and whose number is equal to that of the variables $x, y, z, \ldots$, diminished by the number of equations of condition. Substituting for $x, y, z, \ldots$, their values in terms of the new variables q_i, it is plain that the last equation will take the following form:

$$\frac{d}{dt}\cdot\Sigma_i p_i\delta q_i - \delta(T+\Omega) = 0.$$

We can find the value of p_i without actually making the substitution, from this consideration; since the original equation contains only the variations $\delta x, \delta y, \delta z, \ldots$, without the variations $\delta\frac{dx}{dt}, \delta\frac{dy}{dt}, \delta\frac{dz}{dt}, \ldots$, it follows that, in its transformed state, it should contain only the variations δq_i without the variations $\delta\frac{dq_i}{dt}$.

Then writing q_i' for $\frac{dq_i}{dt}$, the coefficient of $\delta q_i'$ should vanish in the equation

$$\Sigma_i\left(\frac{dp_i}{dt}\delta q_i + p_i\delta q_i'\right) - \delta(T+\Omega) = 0.$$

That is, since Ω does not contain q_i',

$$p_i = \frac{\partial T}{\partial q_i'}.$$

Thus the general formula becomes

$$\frac{d}{dt}\cdot\Sigma_i\left(\frac{\partial T}{\partial q_i'}\delta q_i\right) - \delta(T+\Omega) = 0,$$

Because in this equation the variables q_i are independent, we may equate the coefficient of each δq_i to zero. Thus

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial q'_i}-\frac{\partial(T+\Omega)}{\partial q_i}=0.$$

This is Lagrange's canonical form of the differential equations of motion.

A simpler form may be obtained by substituting the variables p_i for q'_i. By adding to and subtracting from the general formula, the term $\delta\,.\,\Sigma_i(p_i,\,q'_i)$, and writing

$$H=\Sigma_i(p_iq'_i)-T-\Omega,$$

it becomes

$$\Sigma_i\left(\frac{dp_i}{dt}\delta q_i-\frac{dq_i}{dt}\delta p_i\right)+\delta H=0.$$

Equating the coefficients of each variation δq_i and δp_i to zero gives the equations

$$\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i},\quad \frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},$$

which are known as Hamilton's canonical form.

The expression for H can take a simpler shape. From the value of T, it is evident that a certain part of it is independent of the variables q'_i, which may be denoted by T_0, another part T_1, involves the first powers, and a third T_2 involves the squares and products of the same; then $T=T_0+T_1+T_2$. By the theory of homogeneous functions

$$\Sigma_i(p_iq'_i)=\Sigma_i\left(\frac{\partial T}{\partial q'_i}q'_i\right)=T_1+2T_2.$$

Hence, if we write

$$\Omega'=\Omega+T_0,$$

we shall have

$$H=T_2-\Omega'.$$

MEMOIR No. 22.

On the Solution of Cubic and Biquadratic Equations.

(The Analyst, Vol. II, pp. 4–8, 1875.)

In nearly all treatises on algebra, the solution of these equations is presented as accomplished by the aid of analytical artifices, which one seems, by some happy chance, to have stumbled upon. No doubt the processes were found in this manner by the original discoverers, Tartaglia, Cardan and Ferrari. But, for many reasons, it would be better to treat the subject as one demanding invention rather than artifice. The equations can, as it were, be interrogated and compelled to yield up their secrets, if they have any.

To say that an equation is solvable algebraically, is to say that an algebraic expression can be found equivalent to the general root, that is, one involving a finite number of the operations of addition, subtraction, multiplication, division and the extraction of roots of prime degree. If the expression does not involve the last mentioned operation, it is called rational, and if free from the two last, integral.

However complex an algebraic expression involving radicals may be, it is evident that there must be at least one radical which is involved in it rationally. Supposing this to be denoted by $R^{\frac{1}{n}}$, n being a prime integer, it is not difficult to convince one's self that, by the proper reductions, the expression can be exhibited thus:

$$p_0 + p_1 R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ldots + p_{n-1} R^{\frac{n-1}{n}},$$

where $p_0, p_1, \ldots .$, do not involve the radical $R^{\frac{1}{n}}$. With no loss of generality, we can suppose $p_1 = 1$; for if p_1 is not zero, we can multiply the quantity under the radical sign by p_1^n, and then take $(p_1^n R)^{\frac{1}{n}}$ as the radical; and in the contrary case, if p_k is one of the quantities p which is not zero; the simplification can be accomplished by putting $R' = p_k^n R^k$. Then

$$p_0 + R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ldots + p_{n-1} R^{\frac{n-1}{n}}$$

may be regarded as the most general form of an algebraic expression.

Here may be enunciated a general proposition, which, although I am not aware that it has ever been proved, is doubtless true and may be used for purposes of discovery. If an algebraic expression exists, equivalent to the general root of the equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots + g = 0,$$

it can be exhibited in the form given above, n being one of the prime factors of m. Thus the algebraic expression of the root of the general equation of the 5th degree, if it existed, could be presented in the form

$$p_2 + R^{\frac{1}{5}} + p_2 R^{\frac{2}{5}} + p_3 R^{\frac{3}{5}} + p_4 R^{\frac{4}{5}},$$

and that of the 6th degree in either of the two forms

$$p_0 + R^{\frac{1}{3}} + p_2 R^{\frac{2}{3}}, \quad p_0 + R^{\frac{1}{2}}.$$

Solution of Cubic Equations.

According to the foregoing proposition, the root of the general cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

if it has an algebraic expression, must be presented in the form

$$x = p + R^{\frac{1}{3}} + p' R^{\frac{2}{3}}.$$

But, since we suppose that this is an irreducible expression involving radicals, it follows that it must satisfy the given equation, whichever of its three values is attributed to the radical $\sqrt[3]{R}$. Thus, calling either of the imaginary cube roots of unity α, the three roots of the cubic equation must be

$$\begin{aligned} x_1 &= p + R^{\frac{1}{3}} + p' R^{\frac{2}{3}}, \\ x_2 &= p + \alpha R^{\frac{1}{3}} + \alpha^2 p' R^{\frac{2}{3}}, \\ x_3 &= p + \alpha^2 R^{\frac{1}{3}} + \alpha^4 p' R^{\frac{2}{3}}. \end{aligned}$$

The first method that suggests itself for obtaining equations which shall give the values of p, p' and R, is to substitute these expressions in the symmetric functions which are equivalent to the several coefficients a, b, c, viz.,

$$x_1 + x_2 + x_3 = -a, \quad x_1 x_2 + x_2 x_3 + x_3 x_1 = b, \quad x_1 x_2 x_3 = -c.$$

But a simpler proceeding is to employ the three symmetric functions $\Sigma . x$, $\Sigma . x^2$ and $\Sigma . x^3$. Since any cube root, as $\sqrt[3]{R}$ is a root of $x^3 - R = 0$, in which the coefficients denoted above by a and b are each zero, it follows that the sum of the three cube roots of any quantity, as well as the sum of

their squares, is zero. Now, it is plain that if the value of x is raised to the n^{th} power,

$$x^n = A + BR^{\frac{1}{3}} + CR^{\frac{2}{3}},$$

where A, B and C are free from the radical $\sqrt[3]{R}$, and are consequently the same whichever of the three roots x denotes. Thus, since $\Sigma . \sqrt[3]{R} = 0$, $\Sigma . \sqrt[3]{R^2} = 0$, we have

$$\Sigma . x^n = 3A.$$

Thus, for computing the value of $\Sigma . x^n$, we need only the part A which is free from the radical $\sqrt[3]{R}$. In this way we obtain and equate to their known values in terms of the coefficients a, b, c,

$$\begin{aligned} \Sigma . x &= 3p &&= -a, \\ \Sigma . x^2 &= 3(p^2 + 2p'R) &&= a^2 - 2b, \\ \Sigma . x^3 &= 3(p^3 + R + 6pp'R + p'^3R^2) &&= -a^3 + 3ab - 3c. \end{aligned}$$

These equations afford the values of p, p' and R; from the first two

$$p = -\frac{a}{3}, \quad p'R = \frac{a^2 - 3b}{9},$$

and by substitution of these values in the last,

$$R^2 + \frac{2a^3 - 9ab + 27c}{27} R + \left(\frac{a^2 - 3b}{9}\right)^3 = 0,$$

a quadratic equation in R; thus the general cubic admits solution by radicals.

For the sake of brevity, putting

$$A = \frac{a^2 - 3b}{9}, \quad B = -\frac{2a^3 - 9ab + 27c}{54},$$

we have

$$R = B \pm \sqrt{B^2 - A^3},$$

and, as we may take at our option either of the two roots, we have choice of the two expressions for x,

$$x = -\tfrac{1}{3}a + [B + \sqrt{B^2 - A^3}]^{\frac{1}{3}} + A[B + \sqrt{B^2 - A^3}]^{-\frac{1}{3}},$$
$$x = -\tfrac{1}{3}a + [B - \sqrt{B^2 - A^3}]^{\frac{1}{3}} + A[B - \sqrt{B^2 - A^3}]^{-\frac{1}{3}}.$$

The three values of x are obtained by attributing in succession to the single cube root appearing in either of these expressions its three values.

I do not know why almost all algebraists prefer to put the root in the form

$$x = -\tfrac{1}{3}a + \sqrt[3]{[B + \sqrt{B^2 - A^3}]} + \sqrt[3]{[B - \sqrt{B^2 - A^3}]}.$$

It is certainly easier in practice to make a division than an extraction of a cube root; moreover, we are troubled, in the latter form, with the selection of the proper three values out of the nine of which it is susceptible, a difficulty which does not occur in the two former expressions.

Solution of Biquadratic Equations.

An algebraic expression for the root of the general equation of the fourth degree

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

if it exists, can be presented in the form $P + \sqrt{Q}$. And if this denotes one of the roots, another will be $P - \sqrt{Q}$; but since x has four values, it is plain that P and Q must receive each two values. This condition will be fulfilled if we suppose that these quantities, in their turn, similarly to x, are rational functions of a second radical $\sqrt{R}$. Thus we put

$$P = p + \sqrt{R}, \quad Q = q + q'\sqrt{R}.$$

Then we have

$$x = p + \sqrt{R} + \sqrt{q + q'\sqrt{R}}.$$

The four values of x are obtained by giving in succession to the radicals $\sqrt{Q}$ and $\sqrt{R}$ all the values they are, in combination, susceptible of. Thus

$$\begin{aligned}
x_1 &= p + \sqrt{R} + \sqrt{q + q'\sqrt{R}}, \\
x_2 &= p - \sqrt{R} + \sqrt{q - q'\sqrt{R}}, \\
x_3 &= p + \sqrt{R} - \sqrt{q + q'\sqrt{R}}, \\
x_4 &= p - \sqrt{R} - \sqrt{q - q'\sqrt{R}}.
\end{aligned}$$

By substituting these in the four symmetric functions $\Sigma . x$, $\Sigma . x^2$, $\Sigma . x^3$ and $\Sigma . x^4$, equations will be found determining p, q, q' and R. Here again, in computing $\Sigma . x^n$, the radicals all disappear; for, whenever a radical is present with one sign in any root, there is always another root in which it is present with the opposite sign; thus these expressions in pairs cancel each other. Then, in deriving $\Sigma . x^n$, it is necessary to preserve only the terms which are free from radicals. In this way we get

$$\begin{aligned}
\Sigma . x &= 4p && = -a, \\
\Sigma . x^2 &= 4[p^2 + q + R] && = a^2 - 2b, \\
\Sigma . x^3 &= 4[p^3 + 3p(q + R) + 3q'R] && = -a^3 + 3ab - 3c, \\
\Sigma . x^4 &= 4[(p^2 + q + R)^2 + (4p^2 + 12pq' + q'^2)R + 4q(p^2 + R)] && = a^4 - 4a^2b + 4ac + 2b^2 - 4d.
\end{aligned}$$

From which we derive

$$p = -\frac{a}{4}, \quad q + R = \frac{3a^2 - 8b}{16}, \quad q'R = -\frac{a^3 - 4ab + 8c}{32},$$

$$R^3 - \frac{3a^2 - 8b}{16}R^2 + \frac{3a^4 - 16a^2b + 16ac + 16b^2 - 64d}{256}R - \left(\frac{3a^3 - 4ab + 8c}{64}\right)^2 = 0.$$

The last is a cubic in R, which, by the foregoing, is solvable by radicals; hence the general equation of the fourth degree is so solvable. In forming the value of x, we may attribute to R as its value any one of the three roots of this equation. When $a = 0$, the case usually treated, the equations are simpler, viz.,

$$p = 0, \quad q + R = -\tfrac{1}{2}b, \quad q'R = -\tfrac{1}{4}c,$$

$$R^3 + \tfrac{1}{2}bR^2 + \frac{b^2 - 4d}{16}R - \frac{c^2}{64} = 0.$$

If we should attempt to treat the general equation of the fifth degree in the preceding manner, we would be led to equations of higher degrees than the fifth, which must be regarded as a strong argument for the non-existence of an algebraic expression equivalent to the root of the general equation of this degree.

MEMOIR No. 23.

On the Equilibrium of a Bar Fixed at One End Half Way between Two Centers of Force.

(The Analyst, Vol. II, pp. 57–59, 1875.)

"A very small bar of matter is movable about one extremity which is fixed half way between two centers of force attracting inversely as the square of the distance; if l be the length of the bar, and $2a$ the distance between the centers of force, prove that there will be two positions of equilibrium for the bar, or four, according as the ratio of the absolute intensity of the more powerful force to that of the less powerful is or is not greater than $(a + 2l) \div (a - 2l)$: and distinguish between the stable and unstable positions."*

Solution.

Assume the fixed extremity of the bar as the origin of coordinates and the direction of the line joining the two centers of force as that of the axis of x. Then x and y being the coordinates of a material point of the bar, and X and Y the forces acting on it, we have from the well-known equations for the motion of a rigid body

$$S\,\frac{xd^2y - yd^2x}{dt^2}\,dm = S(xY - yX).$$

If M and M' denote the intensities of the forces at the unit of distance, we have

$$X = \frac{M(a-x)\,dm}{[(a-x)^2 + y^2]^{\frac{3}{2}}} - \frac{M'(a+x)\,dm}{[(a+x)^2 + y^2]^{\frac{3}{2}}},$$
$$Y = -\frac{Mydm}{[(a-x)^2 + y^2]^{\frac{3}{2}}} - \frac{M'ydm}{[(a+x)^2 + y^2]^{\frac{3}{2}}}.$$

Introduce polar coordinates, and put

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and since the mass of the bar may be supposed evenly distributed along its length, put $dm = dr$, and take the integration with respect to r between the

* Cambridge Problems for 1845.

limits 0 and l. These substitutions made in the equations of motion, we get

$$\frac{l^2}{3}\frac{d^2\theta}{dt^2} = a\sin\theta\int_0^l\left[-\frac{Mrdr}{[a^2-2ar\cos\theta+r^2]^{\frac{3}{2}}}+\frac{M'rdr}{[a^2+2ar\cos\theta+r^2]^{\frac{3}{2}}}\right].$$

Or, the integration performed,

$$\frac{l^3}{3}\frac{d^2\theta}{dt^2} = -\frac{M\sin\theta}{[a-l\cos\theta+\sqrt{(a^2-2al\cos\theta+l^2)}]\sqrt{(a^2-2al\cos\theta+l^2)}}$$
$$+\frac{M'\sin\theta}{[a+l\cos\theta+\sqrt{(a^2+2al\cos\theta+l^2)}]\sqrt{(a^2+2al\cos\theta+l^2)}}.$$

This differential equation determines θ and thus the position of the bar at any moment. For equilibrium the right member must vanish; thus $\theta=0$, $\theta=\pi$ are two positions of equilibrium. If there are any others, the equation

$$\frac{[a-l\cos\theta+\sqrt{(a^2-2al\cos\theta+l^2)}]\sqrt{(a^2-2al\cos\theta+l^2)}}{[a+l\cos\theta+\sqrt{(a^2+2al\cos\theta+l^2)}]\sqrt{(a^2+2al\cos\theta+l^2)}}=\frac{M}{M'}$$

must be satisfied. But the numerator of the left member of this equation evidently has its minimum value when $\theta=0$, and constantly increases from this point until $\theta=\pi$ when the maximum value is attained. On the other hand, the denominator has its maximum value when $\theta=0$, and constantly diminishes from this point until $\theta=\pi$, when the minimum is attained. From this it is plain that the minimum value of the left member is $\left(\frac{a-l}{a+l}\right)^2$, the maximum value $\left(\frac{a+l}{a-l}\right)^2$, and that the member continually augments in going from first to second. Hence if $\frac{M}{M'}$ lie between $\left(\frac{a-l}{a+l}\right)^2$ and $\left(\frac{a+l}{a-l}\right)^2$, there will be two additional positions of equilibrium, one between $\theta=0$ and $\theta=\pi$, and the other between $\theta=\pi$ and $\theta=2\pi$; in the contrary case there will be none.

When we have nearly $\theta=0$, the differential equation reduces sensibly to

$$\frac{l^2}{3}\frac{d^2\theta}{dt^2}=[-M(a-l)^{-2}+M'(a+l)^{-2}]\sin\theta,$$

and when nearly $\theta=\pi$, to

$$\frac{l^2}{3}\frac{d^2\theta}{dt^2}=[-M(a+l)^{-2}+M'(a-l)^{-2}]\sin\theta.$$

Thus the position of equilibrium when $\theta=0$ is stable or unstable according as $\frac{M}{M'}$ is greater or less than $\left(\frac{a-l}{a+l}\right)^2$, and when $\theta=\pi$, the equilibrium is stable or unstable according as $\frac{M}{M'}$ is less or greater than $\left(\frac{a+l}{a-l}\right)^2$.

The two remaining positions of equilibrium, when they exist, are always unstable, as will be plain from considering the mode of increase of the function of θ which is equivalent to $\frac{d^2\theta}{dt^2}$.

NOTE.

The foregoing solution agrees with the statement of the problem, if we suppose that l is so small that its square may be neglected. It may be added that the preceding expression for $\frac{d^2\theta}{dt^2}$ is complex only because it is necessary to make $\sin\theta$ appear as a factor. If ψ and ψ' denote the angles at the base of the triangle formed by the two centers of force and the extremity of the bar, the differential equation can be written thus

$$\frac{l^3}{3}\frac{d^2\theta}{dt^2} = -2M\sin^2\frac{\psi}{2} + 2M'\sin^2\frac{\psi'}{2}.$$

MEMOIR No. 24.

The Deflection Produced in the Direction of Gravity at the Foot of a Conical Mountain of Homogeneous Density.

(The Analyst, Vol. 11, pp. 119–120, 1875.)

Assume the station as the origin of coordinates, the axis of x being directed toward the center of the base of the mountain, and that of z vertical. Let a be the radius of the base and c the altitude of the mountain. The equation of the mountain's surface is then

$$a^2 (c - z)^2 = c^2 [(a - x)^2 + y^2] .$$

The equation in terms of polar coordinates is obtained by putting

$$x = r \cos\theta \cos\omega , \quad y = r \cos\theta \sin\omega , \quad z = r \sin\theta ,$$

and thus is

$$r = 2ac \frac{c \cos\theta \cos\omega - a \sin\theta}{c^2 \cos^2\theta - a^2 \sin^2\theta} .$$

The element of volume of the mountain may be regarded as a rectangular solid whose sides are dr, $r \cos\theta\, d\omega$, $r d\theta$, and ρ being its density, the element of mass is $\rho\, r^2 \cos\theta\, dr\, d\theta\, d\omega$. Its attraction on the unit of mass at the station is $\rho \cos\theta\, dr\, d\theta\, d\omega$. From the symmetry of the cone it is plain that the component of the mountain's attraction in the direction of the axis of y is zero; and the vertical component which diminishes the intensity of gravity at the station may be neglected. The component in the direction of the axis of x is

$$X = \rho \int\int\int \cos^2\theta \cos\omega\, dr\, d\theta\, d\omega .$$

Integrating with respect to r, the limits are $r = 0$ and $r =$ the value given by the equation of the surface. Thus

$$X = 2ac\rho \int\int \frac{c \cos\theta \cos\omega - a \sin\theta}{c^2 \cos^2\theta - a^2 \sin^2\theta} \cos^2\theta \cos\omega\, d\theta\, d\omega .$$

Next we integrate with respect to ω. As r must be always positive, the limiting values of ω are the two roots of the equation $c \cos\omega = a \tan\theta$. Hence

$$X = 2ac\rho\int\left[\frac{c\cos^3\theta\cos^{-1}\left[\frac{a}{c}\tan\theta\right]}{c^2\cos^2\theta - a^2\sin^2\theta} - \frac{\sin\theta\cos\theta}{\sqrt{c^2\cos^2\theta - a^2\sin^2\theta}}\right]d\theta.$$

The limits of integration are now from $\theta = 0$ to $\theta =$ the value given by the equation $a\tan\theta = c$. The second term within the brackets is integrable, and between the limits is $-\frac{a}{a^2+c^2}$. To simplify the first term, revert to the variable ω, that is, put $a\tan\theta = c\cos\omega$. Then

$$X = 2c\rho\left[\int_0^{\frac{\pi}{2}}\frac{\omega d\omega}{\sin\omega\left[1+\frac{c^2}{a^2}\cos^2\omega\right]^{\frac{3}{2}}} - \frac{a^2}{a^2+c^2}\right].$$

The expression within the brackets is a function of $\frac{c}{a}$, calling it $F\left(\frac{c}{a}\right)$, we have

$$X = 2F\left(\frac{c}{a}\right)c\rho.$$

Now ρ' being the mean density and R the radius of the earth, the force of gravity is

$$g = \frac{4\pi}{3}\rho' R,$$

and δ the deflection of the plumb-line is given by the equation

$$\tan\delta = \frac{X}{g} = \frac{3F\left(\frac{c}{a}\right)}{2\pi}\frac{\rho}{\rho'}\frac{c}{R}.$$

The definite integral

$$\int_0^{\frac{\pi}{2}}\frac{\omega d\omega}{\sin\omega\left[1+\frac{c^2}{a^2}\cos^2\omega\right]^{\frac{3}{2}}},$$

it appears, must be computed by mechanical quadratures.

As an example in illustration, suppose $a = 5$ miles, $c = 2$ miles, $R = 3956$ miles, $\rho = 2.75$ and $\rho' = 5.67$. For evaluating the definite integral, divide the interval between 0 and $\frac{\pi}{2}$ into 9 equal parts; then $h = 10°$ $= 0.1745241$. Compute the value of the function to be integrated multiplied by h for the middle of each of these parts, that is, for $\omega = 5°, 15° \ldots . 115°$. The three values beyond 90° are for the sake of the differences. We get

ω.	Δ_0.	ω.	Δ_0.	ω.	Δ_0.
5°	0.1400956	45°	0.1727216	85°	0.2594408
15	0.1432880	55	0.1893800	95	0.2899632
25	0.1497300	65	0.2094292	105	0.3258781
35	0.1595134	75	0.2327701	115	0.3705285

As the function integrated remains the same when the sign of ω is changed, all the odd orders of differences vanish for the argument $\omega = 0$. Then making $\Delta^{-1} = 0$, for the argument $\omega = 0$, by summing and differencing, we get for the argument $\omega = 90°$,

$$\Delta^{-1} = 1.6563687, \quad \Delta^1 = +0.0305224, \quad \Delta^3 = +0.0015408, \quad \Delta^5 = +0.0007833.$$

Thus the value of the definite integral is

$$1.6563687 + \tfrac{1}{24}(0.0305224) - \tfrac{17}{5760}(0.0015408) + \tfrac{367}{967680}(0.0007833) = 1.6576363.$$

Consequently $F(0.4) = 0.7955673$, and the deflection

$$\delta = 19''.21174.$$

MEMOIR No. 25.

On the Development of the Perturbative Function in Periodic Series.

(The Analyst, Vol. II, pp. 161–180. 1875.)

1. There are two modes of developing this function. In one, the numerical values of the elements involved are employed from the outset, and the results obtained belong only to the special case treated. This mode has been, almost exclusively, followed by Hansen, and is, perhaps, to be recommended when numerical results are chiefly desired. In the other, all the elements are left indeterminate, and thus is obtained a literal development possessing as much generality as possible. Certain investigations, arising from Jacobi's treatment of dynamical equations and Delaunay's method in the lunar theory, have invested the latter mode of development with additional interest, and with it we shall be exclusively engaged in this article.

In Liouville's Journal for 1860, M. Puiseux has given us two memoirs on this subject, in which appears the general term of this function, but his formulas seem susceptible of modifications which would render them much simpler. More recently, in the volume of the same journal for 1873, M. Bourget has presented the development in a more concise form by employing the Besselian functions, but as he discards the use of the functions $b_s^{(i)}$, his formulas on this account are more complex. It is hoped, that, even if the expressions, given hereafter, are deemed too cumbrous for practical use, they may still possess some interest from a theoretical point of view.

2. It is known that if we have a function S of a variable ζ, which is never infinite, and such that the relation

$$\text{function}\,(\zeta + 2i\pi) = \text{function}\,(\zeta)$$

is satisfied for all integral values of i both positive and negative, it can be developed in a series of the form

$$\Sigma_i \,.\, (K_i^{(c)} \cos i\zeta + K_i^{(s)} \sin i\zeta)\,,$$

in which i denotes a positive integer; and that, in the cases where this series is infinite, it is convergent.

In general, the handling of periodic series is easier if we introduce imaginary exponentials in the place of the circular functions. Thus, ε denoting the base of natural logarithms, we shall put $z = \varepsilon^{\zeta\sqrt{-1}}$, whence

$$2\cos\zeta = z + z^{-1}, \qquad 2\cos i\zeta = z^i + z^{-i},$$
$$2\sqrt{(-1)}\sin\zeta = z - z^{-1}, \quad 2\sqrt{(-1)}\sin i\zeta = z^i - z^{-i},$$
$$z = \cos\zeta + \sqrt{(-1)}\sin\zeta, \quad z^i = \cos i\zeta + \sqrt{(-1)}\sin i\zeta.$$

The above theorem then comes to the same thing as to say that S is developable in a series of the form

$$\Sigma_i . C_i z^i,$$

where the summation is extended to negative as well as positive values of i. The coefficients K are given in terms of the coefficients C by the equations

$$K_i^{(c)} = C_i + C_{-i}, \quad K_i^{(s)} = (C_i - C_{-i})\sqrt{-1},$$

except the case where $i = 0$, when $K_0^{(c)} = C_0$. It will be seen that when S is real, C_i is a complex number $a + b\sqrt{-1}$, and C_{-i}, its conjugate $a - b\sqrt{-1}$, which renders the coefficients K real, as they should be.

The integral

$$\int z^i d\zeta = \int (\cos i\zeta + \sqrt{(-1)}\sin i\zeta)\, d\zeta,$$

taken between the limits 0 and 2π, vanishes in all cases except when $i = 0$, when its value is 2π. Hence any function, capable of expansion in a series of positive and negative integral powers of z, integrated with respect to ζ between these limits, gives, as the result, 2π times the coefficient of z^0 in its expansion. And as the coefficient of z^0 in the function Sz^{-i} is evidently C_i, we have

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} Sz^{-i} d\zeta.$$

This equation holds for all values of i, negative as well as positive, zero included.

3. Let us now suppose that ζ denotes the mean anomaly of a planet, and let u be the eccentric anomaly, connected with the former by the equation, e being the eccentricity,

$$u - e\sin u = \zeta.$$

In like manner as for ζ, we introduce the imaginary exponential $s = \varepsilon^{u\sqrt{-1}}$. Then the last equation can be written

$$\varepsilon^{(u - e\sin u)\sqrt{-1}} = \varepsilon^{\zeta\sqrt{-1}},$$

and, by the introduction of the variables s and z, this becomes

$$s\varepsilon^{-\frac{e}{2}\left(s-\frac{1}{s}\right)}=z\,,$$

which is the transcendental equation connecting s and z. We have

$$d\zeta=(1-e\cos u)\,du=\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]du.$$

Substituting these values in the equation giving the value of C_i, and noticing that, as ζ and u both take the values 0 and 2π together, the limits of integration, when u is the independent variable, are the same as for ζ, we get

$$C_i=\frac{1}{2\pi}\int_0^{2\pi}Ss^{-i}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]du.$$

But, from what precedes, we conclude that the coefficient of s^i in the expansion of any function W, according to positive and negative powers of s, is

$$\frac{1}{2\pi}\int_0^{2\pi}Ws^{-i}du.$$

Thus, from the foregoing expression for C_i, we derive the following proposition:

i being a positive or negative integer or zero, the coefficient of z^i, in the development of S, according to the powers of z, is equal to that of s^i in the development of

$$S\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right],$$

according to the powers of s.

As most of the functions S, which are presented by astronomy for development in powers of z, are quite readily expanded in powers of s, this theorem is of much use. Another form can be given to it. For we have, integrating by parts

$$\begin{aligned}\int Sz^{-i}d\zeta&=-\sqrt{(-1)}\int Sz^{-(i+1)}dz\\&=\frac{\sqrt{-1}}{i}Sz^{-i}-\frac{\sqrt{-1}}{i}\int\frac{dS}{dz}z^{-i}dz.\end{aligned}$$

Taking the integrals between the limits $\zeta=0$ and $\zeta=2\pi$, we get

$$\begin{aligned}C_i&=-\frac{\sqrt{-1}}{2i\pi}\int\frac{dS}{ds}z^{-i}ds\\&=\frac{1}{2i\pi}\int_0^{2\pi}\frac{dS}{ds}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}s^{-(i-1)}du.\end{aligned}$$

Whence we conclude this proposition:

The coefficient of z^i in the development of S according to the powers of z is equal to that of s^{i-1} in the development of

$$\frac{1}{i}\cdot\frac{dS}{ds}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

according to the powers of s.

This theorem however is not applicable when $i=0$.

4. We shall often have occasion for the expansion of the function

$$\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

in powers of s; let us, for simplicity, put $\lambda = \frac{ie}{2}$, and

$$\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \Sigma_i\,.\,J_\lambda^{(i)} s^i.$$

We have

$$\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \left[1+\frac{\lambda s}{1}+\frac{\lambda^2 s^2}{1\,.\,2}+\frac{\lambda^3 s^3}{1\,.\,2\,.\,3}+\cdots\right]\cdot\left[1-\frac{\lambda}{s}+\frac{1}{1\,.\,2}\frac{\lambda^2}{s^2}-\frac{1}{1\,.\,2\,.\,3}\frac{\lambda^3}{s^3}+\cdots\right],$$

whence we conclude that

$$J_\lambda^{(i)} = \frac{\lambda^i}{1\,.\,2\ldots i}\left[1-\frac{\lambda^2}{1\,.\,(i+1)}+\frac{\lambda^4}{1\,.\,2\,(i+1)(i+2)}-\cdots\right].$$

This series is not applicable when i is negative; but if, in the function $\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}}$, we substitute $\frac{1}{s}$ for s, and change the sign of λ, the function remains unchanged, hence

$$\Sigma_i\,.\,J_\lambda^{(i)} s^i = \Sigma_i\,.\,J_{-\lambda}^{(i)} s^{-i},$$

and, consequently,

$$J_\lambda^{(-i)} = J_{-\lambda}^{(i)} = (-1)^i J_\lambda^{(i)},$$

by which the values of these functions for negative values of i can be derived from those in which i is positive. These functions are known as the Besselian. By putting

$$T_i = 1-\frac{\lambda^2}{1\,.\,(i+1)}+\frac{\lambda^4}{1\,.\,2\,(i+1)(i+2)}-\cdots$$

one will have no difficulty in deducing the equation

$$T_{i-1} = T_i - \frac{\lambda^2}{i\,(i+1)}\,T_{i+1}.$$

5. We come now to the more complex function S of two variables ζ and ζ'; it is known that when this is never infinite and is such that

$$\text{function}\,(\zeta + 2i\pi,\ \zeta' + 2i'\pi) = \text{function}\,(\zeta,\ \zeta')$$

it can be developed in a series of the form

$$\Sigma_{i,i'}\,[K^{(c)}_{i,i'} \cos(i\zeta + i'\zeta') + K^{(s)}_{i,i'} \sin(i\zeta + i'\zeta')],$$

where to one of the quantities i and i', we need assign only positive integral values, but to the other both positive and negative values. If we adopt another imaginary exponential $z' = \varepsilon^{\zeta'\sqrt{-1}}$, this is the same as saying that

$$S = \Sigma_{i,i'}\,.\,C_{i,i'} z^i z'^{i'},$$

where the summation is extended to all integral values positive and negative for i and i'. Since we have

$$\begin{aligned} z^i z'^{i'} &= (\cos i\zeta + \sqrt{(-1)}\sin i\zeta)(\cos i'\zeta' + \sqrt{(-1)}\sin i'\zeta') \\ &= \cos(i\zeta + i'\zeta') + \sqrt{(-1)}\sin(i\zeta + i'\zeta'), \end{aligned}$$

the relations, which connect the coefficients K with the coefficients C, are

$$\begin{aligned} K^{(c)}_{i,i'} &= C_{i,i'} + C_{-i,-i'}, \\ K^{(s)}_{i,i'} &= (C_{i,i'} - C_{-i,-i'})\sqrt{-1}, \end{aligned}$$

unless i and i' are both zero, when

$$K^{(c)}_{0,0} = C_{0,0}.$$

A course of reasoning, similar to that in the case of one variable, establishes that

$$C_{i,i'} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} S z^{-i} z'^{-i'} d\zeta\, d\zeta',$$

which holds for all integral values of i and i', positive, negative and zero.

6. Supposing that ζ' denotes the mean anomaly of a second planet, whose eccentricity and eccentric anomaly are respectively e' and u', we have

$$u' - e' \sin u' = \zeta',$$

and by the adoption of the imaginary exponential $s' = \varepsilon^{u'\sqrt{-1}}$, this is transformed into

$$s'\varepsilon^{-\frac{e'}{2}\left(s' - \frac{1}{s'}\right)} = z'.$$

It is not difficult to see that we have the following theorem:

The coefficient of $z^i z'^{i'}$ in the development of S, according to the powers of z and z', is equal to that of $s^i s'^{i'}$ in the development of

$$S\varepsilon^{\frac{ie}{2}\left(s - \frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right] . \varepsilon^{\frac{i'e'}{2}\left(s' - \frac{1}{s'}\right)}\left[1 - \frac{e'}{2}\left(s' + \frac{1}{s'}\right)\right],$$

according to the powers of s and s'.

7. After these preliminaries relative to the general development of functions in periodic series, we come to the matter more immediately engaging our attention. The perturbative function for the action of a planet, whose mass is m', on another, whose mass is m, is usually written

$$R = m'\left[\frac{1}{\Delta} - \frac{r\cos\psi}{r'^2}\right],$$

and that for the action of m on m'

$$R_1 = m\left[\frac{1}{\Delta} - \frac{r'\cos\psi}{r^2}\right],$$

where Δ denotes their mutual distance, ψ their angular distance as seen from the sun, and r and r' their radii vectors. The problem proposed is then to develop these two functions in series whose general term is of the form $C_{i,i'}z^iz'^{i'}$. To this end it seems better to discuss the two portions of the general perturbative function, $\frac{1}{\Delta}$ and $-\frac{r\cos\psi}{r'^2}$, separately, and not, as most investigators, attempt, by a particular notation, to combine, in a whole, these two parts. Thus, in developing $\frac{1}{\Delta}$, we shall have the term common to both functions, and may suppose that r' denotes the radius vector which belongs to the planet more distant from the sun. But, in treating the second part, we shall suppose that r' belongs to the disturbing planet. The following equations are well known:

$$\Delta^2 = r'^2 - 2rr'\cos\psi + r^2,$$
$$\cos\psi = \cos(v+\Pi)\cos(v'+\Pi') + \cos I\sin(v+\Pi)\sin(v'+\Pi'),$$
$$= \cos(v - v' + \Pi - \Pi') - 2\sin^2\tfrac{1}{2}I\sin(v+\Pi)\sin(v'+\Pi'),$$

where v and v' are the true anomalies, and Π and Π' are the angular distances of the perihelia from either point of intersection of the planes of the orbits, and I is their mutual inclination.

8. Attending then, in the first place, to the development of $\frac{1}{\Delta}$, we have to notice what are the conditions under which this quantity can be developed in powers of z and z'. In the case of two elliptic orbits, the only one we shall consider here, it is plain that $\frac{1}{\Delta}$ is always finite and continuous, provided the orbits have no point in common. Here we must make two cases according as the value of sin I is not or is zero. In the first case it is evident that the orbits can meet only on the line of intersection of their planes. Hence, p and p' denoting their semi-parameters, there will be

two, one or no points in common, according as two, one or none of the equations,

$$p'(1+e'\cos\Pi')^{-1}=p(1+e\cos\Pi)^{-1},$$
$$p'(1-e'\cos\Pi')^{-1}=p(1-e\cos\Pi)^{-1},$$

are satisfied. In the second case, where the orbits lie in the same plane, there will be two intersections or none, according as the equation

$$p'[1+e'\cos(\lambda-\omega')]^{-1}=p[1+e\cos(\lambda-\omega)]^{-1},$$

λ being the unknown quantity and ω and ω' the longitudes of the perihelia, admits real or imaginary roots. If we put

$$pe'\cos\omega'-p'e\cos\omega=A\cos\alpha,$$
$$pe'\sin\omega'-p'e\sin\omega=A\sin\alpha,$$

this equation takes the form

$$A\cos(\lambda-\alpha)=p'-p.$$

The roots of this are imaginary when

$$(p'-p)^2>p^2e'^2-2pp'ee'\cos(\omega-\omega')+p'^2e^2.$$

9. If we put

$$P=r'^2-2rr'\cos(v-v'+\Pi-\Pi')+r^2,$$
$$Q=4\sin^2\tfrac{1}{2}I\,.\,r\sin(v+\Pi)\,.\,r'\sin(v'+\Pi')$$

we have

$$\Delta^2=P+Q,$$
$$\frac{1}{\Delta}=[P+Q]^{-\frac{1}{2}}$$
$$=P^{-\frac{1}{2}}-\tfrac{1}{2}P^{-\frac{3}{2}}Q+\tfrac{1}{2}\,.\,\tfrac{3}{4}P^{-\frac{5}{2}}Q^2-\ldots.,$$

a series we shall denote thus

$$\frac{1}{\Delta}=\sum_{k=0}^{k=\infty}\cdot(-1)^k\frac{1\,.\,3\ldots(2k-1)}{2\,.\,4\ldots 2k}P^{-\frac{2k+1}{2}}Q^k.$$

10. In order that this development of $\frac{1}{\Delta}$ in a series of ascending powers of Q, or, if one likes, of $\sin^2\frac{1}{2}I$, may be legitimate, it is necessary that the elements of the orbits should be such that the numerical value of $\frac{Q}{P}$ should be always less than unity. P is the square of the distance of the two planets after the plane of the orbit of one has been brought into coincidence with the plane of the other by revolving it about the line of

intersection of the two planes. Taking then a system of rectangular axes passing through the center of the sun, and directing the axis of x along the line of intersection, it is plain the equations of the orbits may be written

$$\sqrt{(x^2+y^2)} + \alpha x + \beta y = p,$$
$$\sqrt{(x'^2+y'^2)} + \alpha' x' + \beta' y' = p',$$

α, β, α', β' being constants. And the variables x, y, x', y' satisfying these equations, the question depends on the finding of the values of them which render the expression

$$D = \frac{yy'}{(x-x')^2 + (y-y')^2}$$

a maximum or a minimum. According to the known theory of maxima and minima, the equations, which, in combination with the equations of the orbits, give these values, are

$$-2D(x-x') + \mu\left[\frac{x}{\sqrt{(x^2+y^2)}} + \alpha\right] = 0,$$
$$2D(x-x') + \mu'\left[\frac{x'}{\sqrt{(x'^2+y'^2)}} + \alpha'\right] = 0,$$
$$y' - 2D(y-y') + \mu\left[\frac{y}{\sqrt{(x^2+y^2)}} + \beta\right] = 0,$$
$$y + 2D(y-y') + \mu'\left[\frac{y'}{\sqrt{(x'^2+y'^2)}} + \beta'\right] = 0,$$

where μ and μ' are the multipliers of the partial derivatives of the two equations of condition. A complete investigation of this question would be conducted in the following manner. Eliminate from the seven equations last given the six quantities x, y, x', y', μ, μ'; the result will be an algebraical equation determining the unknown D. Having derived the Sturmian functions of this, one will ascertain by the substitution of the values $D = \frac{1}{4\sin^2\frac{1}{2}I}$, $D = +\infty$, and again of $D = -\frac{1}{4\sin^2\frac{1}{2}I}$, $D = -\infty$, whether any roots lie between these limits; if none, $\frac{1}{\Delta}$ can be expanded in a series of ascending powers of $\sin^2\frac{1}{2}I$, in the contrary case not. In this way we shall arrive at the condition or conditions necessary and sufficient for the legitimacy of this expansion.

11. This procedure would doubtless lead to very complicated formulas, hence we are obliged to pass over it. However, equations can be readily got, which, by a tentative process, afford the maximum and minimum values of D. Multiply the four equations last given respectively by x, x', y, y'

and add the resulting equations, having regard to the equations of the orbits and the value of D; we thus arrive at the simple relation

$$p\mu + p'\mu' = 0.$$

Putting, for simplicity, $x = r \cos \theta$, $x' = r' \cos \theta'$, the addition of the first and second of the same group of four equations gives

$$\mu (\cos \theta + a) + \mu' (\cos \theta' + a') = 0.$$

By combining this with the preceding is obtained

$$\frac{\cos \theta' + a'}{p'} = \frac{\cos \theta + a}{p}.$$

Again the addition of the same equations, multiplied severally by x, $-x'$, y, $-y'$, gives the equation

$$2D (r'^2 - r^2) = p'\mu' - p\mu.$$

Dividing the left member of this by $2D (x' - x)$, and the terms of the right member by its equivalents derived from the first and second equations, we get

$$\frac{r'^2 - r^2}{x' - x} = \frac{p'}{\cos \theta' + a'} + \frac{p}{\cos \theta + a},$$

or

$$\frac{r' \cos \theta' - r \cos \theta}{r'^2 - r^2} = \frac{\cos \theta + a}{2p}.$$

This and the equation

$$\frac{\cos \theta' + a'}{p'} = \frac{\cos \theta + a}{p}$$

determine the values of the variables θ and θ' which render D a maximum or minimum. When the orbits are nearly circular these values are in the neighborhood of $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. When both orbits are circles the solution is very simple, and we find, in order that the development may be legitimate, we must have

$$\sin \frac{I}{2} < \frac{a' - a}{2\sqrt{(aa')}},$$

a and a' being the mean distances of the planets from the sun.

12. Assuming that this development is legitimate, we have to develop $P^{-\frac{2k+1}{2}} Q^k$ in terms of s and s'. We have

$$r\cos v = a(\cos u - e) = \frac{a}{2}\left(s + \frac{1}{s} - 2e\right),$$
$$r\sin v = a\sqrt{(1-e^2)}\sin u = \frac{a}{2\sqrt{-1}}\sqrt{(1-e^2)}\left(s - \frac{1}{s}\right),$$

whence

$$r\cos v + r\sin v.\sqrt{-1} = r\varepsilon^{v\sqrt{-1}} = a\left[\frac{1+\sqrt{(1-e^2)}}{2}s + \frac{1-\sqrt{(1-e^2)}}{2}\frac{1}{s} - e\right],$$

and by putting

$$\frac{1+\sqrt{(1-e^2)}}{2} = \eta, \quad \frac{e}{1+\sqrt{(1-e^2)}} = \omega,$$

we get

$$r\varepsilon^{v\sqrt{-1}} = a\eta s\left(1 - \frac{\omega}{s}\right)^2.$$

And the value of $r\varepsilon^{-v\sqrt{-1}}$ is evidently obtained by substituting in this $\frac{1}{s}$ for s, hence

$$r\varepsilon^{-v\sqrt{-1}} = a\eta\frac{1}{s}(1-\omega s)^2.$$

From these two equations may be derived

$$r = a\eta(1-\omega s)\left(1 - \frac{\omega}{s}\right),$$
$$\varepsilon^{v\sqrt{-1}} = \frac{s-\omega}{1-\omega s}.$$

Writing γ for $\Pi - \Pi'$, we have

$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \left[1 - 2\frac{r}{r'}\cos(v - v' + \gamma) + \frac{r^2}{r'^2}\right]^{-\frac{2k+1}{2}}.$$

The right member of this is developable in a series of integral powers of the exponential $\varepsilon^{(v-v'+\gamma)\sqrt{-1}}$ when $\frac{r}{r'}$ is always less than unity. This condition is fulfilled when we have $a(1+e) < a'(1-e')$. Writing g for $\varepsilon^{\gamma\sqrt{-1}}$, let

$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} . B^{(j)}_{\frac{2k+1}{2}}\varepsilon^{j(v-v'+\gamma)\sqrt{-1}}$$
$$= \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} . B^{(j)}_{\frac{2k+1}{2}}\left(\frac{s-\omega}{1-\omega s}\right)^j\left(\frac{s'-\omega'}{1-\omega' s'}\right)^{-j}g^j.$$

$B^{(j)}_{\frac{2k+1}{2}}$ is the same function of $\frac{r}{r'}$ that Laplace's $b^{(j)}_{\frac{2k+1}{2}}$ is of $\frac{a}{a'} = \alpha$. The approximate value of $\frac{r}{r'}$ being α, any function of $\frac{r}{r'}$ can be expanded in a series of ascending powers of $\frac{r}{r'} - \alpha$ by Taylor's Theorem. And as we have

$$\frac{r}{r'} - a = a \left\{ \frac{\eta\,(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'\,(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - 1 \right\},$$

consequently,

$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty} \frac{1}{n!}\, a^n \frac{d^n b^{(j)}_{\frac{2k+1}{2}}}{da^n} \left\{ \frac{\eta\,(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'\,(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - 1 \right\}^n,$$

n being an integer, and $n!$ denoting the product of all integers up to n inclusive, it being understood that $0! = 1$. Expanding the last factor of this expression by the binomial theorem, and employing the notation $[i, j]$ for the coefficient of x^j in the expansion of $(1+x)^i$, we have, p being an integer,

$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty} \sum_{p=0}^{p=n} (-1)^{n-p} \frac{[n, p]}{n!}\, a^n \frac{d^n b^{(j)}_{\frac{2k+1}{2}}}{da^n} \left\{ \frac{\eta\,(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'\,(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} \right\}^p.$$

13. In the next place the development of Q in terms of s and s' must be formed. We have

$$r \sin(v + \Pi) = \frac{1}{2\sqrt{-1}} \left[r \varepsilon^{(v+\Pi)\sqrt{-1}} - r \varepsilon^{-(v+\Pi)\sqrt{-1}}\right],$$

$$r' \sin(v' + \Pi') = \frac{1}{2\sqrt{-1}} \left[r' \varepsilon^{(v'+\Pi')\sqrt{-1}} - r' \varepsilon^{-(v'+\Pi')\sqrt{-1}}\right],$$

and putting

$$\Pi + \Pi' = \theta, \qquad h = \varepsilon^{\theta\sqrt{-1}},$$

we find

$$Q = -aa'\eta\eta' \sin^2\frac{I}{2} \cdot \left[s\left(1-\frac{\omega}{s}\right)^2 - \frac{1}{s}(1-\omega s)^2 g^{-1} h^{-1} \right] \times \left[s'\left(1-\frac{\omega'}{s'}\right)^2 h - \frac{1}{s'}(1-\omega' s')^2 g \right].$$

Raising this expression to the k^{th} power, and multiplying by

$$r'^{-(2k+1)} = \left[a'\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)\right]^{-(2k+1)},$$

we find that the part of $r'^{-(2k+1)} Q^k$ which has $h^{i'''}$ as a factor is

$$\frac{1}{a'} a^k \eta^k \eta'^{-(k+1)} \sin^{2k}\frac{I}{2} \sum_{n'=0}^{n'=k-i'''} (-1)^{i'''} [k, n'][k, k-i'''-n']$$
$$\times\, s^{2i'''-k+2n'} (1-\omega s)^{2k-2i'''-2n'} \left(1-\frac{\omega}{s}\right)^{2i'''+2n'}$$
$$\times\, s'^{k-2n'} (1-\omega' s')^{2n'-2k-1} \left(1-\frac{\omega'}{s'}\right)^{-2n'-1} g^{i'''-k+2n'} h^{i'''}.$$

14. We are now in the possession of all the developments necessary for exhibiting the function $\frac{1}{\Delta}$ in terms of s and s'. In order to obtain the part of this function which has $g^{i''}h^{i'''}$ for a factor, we must put, in the formulas of §12,

$$j = i'' - i''' + k - 2n',$$

and the chief operation here is the addition of the exponents of the quantities s, $1 - \omega s$, $1 - \frac{\omega}{s}$, and the similar functions of s' which are found in the three formulas for $(r'^{-2}P)^{-\frac{2k+1}{2}}$, $B^{(j)}_{\frac{2k+1}{2}}$ and $r'^{-(2k+1)}Q^k$. For brevity we will write

$$[k] = \frac{1.3\ldots(2k-1)}{2.4\ldots.2k}.$$

Then the part of $\frac{1}{\Delta}$, which has $g^{i''}h^{i'''}$ for a factor, is

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\ \sum_{n'=0}^{n'=k-i'''}\ \sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][k,n'][k,k-i'''-n'][n,p]}{n!}$$
$$\times\, a^{k+n}\frac{d^n b^{(i''-i'''+k-2n')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\,\eta^{k+p}s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$
$$\times\, \eta'^{-k-p-1}s'^{-i''+i'''}(1-\omega' s')^{-k-p-1+i''+i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''} g^{i''}h^{i'''}.$$

We observe that in this expression the summation with respect to n' affects only the integral coefficients $[k, n']$, $[k, k-i'''-n']$ and the upper index of the quantity b, hence if a new function of α is assumed, which is a linear function of the b's, and such that

$$\mathsf{B}^{(i'',i''')}_{\frac{2k+1}{2}} = \sum_{n'=0}^{n'=k-i'''}[k,n'][k,k-i'''-n']\,b^{(i''-i'''+k-2n')}_{\frac{2k+1}{2}},$$

it will take the following simpler form:

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][n,p]}{n!}\,a^{k+n}\frac{d^n\mathsf{B}^{(i'',i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}$$
$$\times\, \eta^{k+p}s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$
$$\times\, \eta'^{-k-p-1}s'^{-i''+i'''}(1-\omega' s')^{-k-p-1+i''-i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''} g^{i''}h^{i'''}.$$

15. In order to get the coefficient of $z^i z'^{i'}$ in the expansion of $\frac{1}{\Delta}$, according to the foregoing investigation, we must multiply the preceding expression by

$$\frac{rr'}{aa'}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)+\frac{i'e'}{2}\left(s'-\frac{1}{s'}\right)}.$$

Hence if for brevity we adopt the functional notation

$$S\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \eta^i s^j (1-\omega s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j} \varepsilon^{\frac{ke}{2}\left(s-\frac{1}{s}\right)},$$

$$S'\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \eta'^i s'^j (1-\omega' s')^{i-j}\left(1-\frac{\omega'}{s'}\right)^{i+j} \varepsilon^{\frac{ke'}{2}\left(s'-\frac{1}{s'}\right)},$$

the coefficient of $z^i z^{i'} g^{i''} h^{i'''}$ in $\frac{1}{\Delta}$ will be equal to the coefficient of $s^i s'^{i'}$ in

$$\frac{1}{2a'} \sum_{k=i'''}^{k=\infty} \sum_{n=0}^{n=\infty} \sum_{p=0}^{p=n} (-1)^{k-i''+n-p} \frac{[k][n,p]}{n!}$$
$$\times\, a^{k+n} \frac{d^n \mathrm{B}^{(i'',\, i''')}_{\frac{2k+1}{2}}}{da^n} \sin^{2k}\frac{I}{2} \,.\, S\begin{pmatrix} k+p \\ i''+i''' \\ i \end{pmatrix} . \, S'\begin{pmatrix} -(k+p+1) \\ -(i''-i''') \\ i' \end{pmatrix}.$$

If then the coefficient of s^i in the expansion of S is denoted by E followed by the same indices, and the coefficient of $s'^{i'}$ in the expansion of S' by E' in like manner, E will be a function of e only, and E' a function of e' only; and, it being understood that each argument is taken but once, that is, the negative of the argument is not considered, the coefficient of

$$\cos(i\zeta + i'\zeta' + i''\gamma + i'''\theta)$$

in the expansion of $\frac{1}{\Delta}$ is expressed thus

$$\frac{1}{a'} \sum_{k=i'''}^{k=\infty} \sum_{n=0}^{n=\infty} \sum_{p=0}^{p=n} (-1)^{k-i'''+n-p} \frac{[k][n,p]}{n!}$$
$$\times\, a^{k+n} \frac{d^n \mathrm{B}^{(i'',\, i''')}_{\frac{2k+1}{2}}}{da^n} \sin^{2k}\frac{I}{2} \,.\, \mathrm{E}\begin{pmatrix} k+p \\ i''+i''' \\ i \end{pmatrix} . \, \mathrm{E}'\begin{pmatrix} -(k+p+1) \\ -(i''-i''') \\ i' \end{pmatrix}.$$

As in this formula, k ought to be a positive integer, it will prevent embarrassment, if the arguments are so taken that i''' may not be negative. In the case where i, i', i'' and i''' are all zero, the expression must be divided by 2.

16. Thus we have arrived at an expression for the general coefficient involving only three signs of summation; and it may be remarked that all the coefficients are exhibited in precisely similar forms. Thus, to pass from one argument to another, we have only to make the suitable changes in the two lower indices of the functions E and E' and in the upper indices of B, and commence the summation with reference to k with the new value of i''' instead of the old. Hence, from this expression, we can write out a scheme or blank form, which, when the indices proper to the argument are filled in, will be the coefficient of the cosine of it in the expansion of $\frac{1}{\Delta}$. Such

a blank form is written below; the indices i'' and i''' are omitted from B, and the two lower indices from E and E′, and the upper indices of these quantities, for the sake of facility in writing, are placed to the right and at the foot. The factor $\frac{1}{a'}$, common to the whole expression, is also omitted, so that the formula gives the coefficient in the expansion of $\frac{a'}{\Delta}$. In making use of it, one must commence at the portion which has $\sin^{2i'''}\frac{1}{2}I$ for a factor, all the preceding parts being supposed to be suppressed. It is hoped that a sufficient number of terms have been written to render the law evident, so that they may be continued as far as desired.

$$
\begin{aligned}
& b_{\frac{1}{2}} E_0 E'_{-1} \\
& -\frac{1}{1} a \frac{db_{\frac{1}{2}}}{da}\left[E_0 E'_{-1} - E_1 E'_{-2}\right] \\
& +\frac{1}{1.2} a^2 \frac{d^2 b_{\frac{1}{2}}}{da^2}\left[E_0 E'_{-1} - 2E_1 E'_{-2} + E_2 E'_{-3}\right] \\
& -\frac{1}{1.2.3} a^3 \frac{d^3 b_{\frac{1}{2}}}{da^3}\left[E_0 E'_{-1} - 3E_1 E'_{-2} + 3E_2 E'_{-3} - E_3 E'_{-4}\right] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
+\frac{1}{2}\sin^2\frac{I}{2}\Big\{ & \quad a B_{\frac{3}{2}} E_1 E'_{-2} \\
& -\frac{1}{1} a^2 \frac{dB_{\frac{3}{2}}}{da}\left[E_1 E'_{-2} - E_2 E'_{-3}\right] \\
& +\frac{1}{1.2} a^3 \frac{d^2 B_{\frac{3}{2}}}{da^2}\left[E_1 E'_{-2} - 2E_2 E'_{-3} + E_3 E'_{-4}\right] \\
& -\frac{1}{1.2.3} a^4 \frac{d^3 B_{\frac{3}{2}}}{da^3}\left[E_1 E'_{-2} - 3E_2 E'_{-3} + 3E_3 E'_{-4} - E_4 E'_{-5}\right] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Big\} \\
+\frac{1.3}{2.4}\sin^4\frac{I}{2}\Big\{ & \quad a^2 B_{\frac{5}{2}} E_2 E'_{-3} \\
& -\frac{1}{1} a^3 \frac{dB_{\frac{5}{2}}}{da}\left[E_2 E'_{-3} - E_3 E'_{-4}\right] \\
& +\frac{1}{1.2} a^4 \frac{d^2 B_{\frac{5}{2}}}{da^2}\left[E_2 E'_{-3} - 2E_3 E'_{-4} + E_4 E'_{-6}\right] \\
& -\frac{1}{1.2.3} a^6 \frac{d^3 B_{\frac{5}{2}}}{da^3}\left[E_2 E'_{-3} - 3E_3 E'_{-4} + 3E_4 E'_{-5} - E_6 E'_{-6}\right] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Big\} \\
+\frac{1.3.5}{2.4.6}\sin^6\frac{I}{2}\Big\{ & \quad a^3 B_{\frac{7}{2}} E_3 E'_{-4} \\
& -\frac{1}{1} a^4 \frac{dB_{\frac{7}{2}}}{da}\left[E_3 E'_{-4} - E_4 E'_{-5}\right] \\
& +\frac{1}{1.2} a^6 \frac{d^2 B_{\frac{7}{2}}}{da^3}\left[E_3 E'_{-4} - 2E_4 E'_{-5} + E_5 E'_{-6}\right] \\
& -\frac{1}{1.2.3} a^6 \frac{d^3 B_{\frac{7}{2}}}{da^3}\left[E_3 E'_{-4} - 3E_4 E'_{-5} + 3E_5 E'_{-6} - E_6 E'_{-7}\right] \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \Big\} \\
+ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

For illustration, let it be desired to obtain the coefficient of

$$\cos(2\zeta - 5\zeta' + 2\gamma),$$

from which arises the larger part of the great inequality of Jupiter and Saturn; we have only to imagine that the lower indices $\begin{pmatrix} \cdot \\ 2 \\ 2 \end{pmatrix}$ are everywhere applied to E, and the indices $\begin{pmatrix} \cdot \\ -2 \\ -5 \end{pmatrix}$ to E', the indices (2, 0) to B; and as we have $i''' = 0$, we suppress nothing.

17. The quantities B are very simply expressed in terms of the b's. The following are all that are needed when terms of the eighth order with respect to the inclination of the orbits are neglected.

$$\begin{aligned}
\mathsf{B}_{\frac{1}{2}}^{(i,0)} &= b_{\frac{1}{2}}^{(i)}, \\
\mathsf{B}_{\frac{3}{2}}^{(i,0)} &= b_{\frac{3}{2}}^{(i+1)} + b_{\frac{3}{2}}^{(i-1)}, \\
\mathsf{B}_{\frac{3}{2}}^{(i,1)} &= b_{\frac{3}{2}}^{(i)}, \\
\mathsf{B}_{\frac{5}{2}}^{(i,0)} &= b_{\frac{5}{2}}^{(i+2)} + 4b_{\frac{5}{2}}^{(i)} + b_{\frac{5}{2}}^{(i-2)}, \\
\mathsf{B}_{\frac{5}{2}}^{(i,1)} &= 2b_{\frac{5}{2}}^{(i+1)} + 2b_{\frac{5}{2}}^{(i-1)}, \\
\mathsf{B}_{\frac{5}{2}}^{(i,2)} &= b_{\frac{5}{2}}^{(i)}, \\
\mathsf{B}_{\frac{7}{2}}^{(i,0)} &= b_{\frac{7}{2}}^{(i+3)} + 9b_{\frac{7}{2}}^{(i+1)} + 9b_{\frac{7}{2}}^{(i-1)} + b_{\frac{7}{2}}^{(i-3)}, \\
\mathsf{B}_{\frac{7}{2}}^{(i,1)} &= 3b_{\frac{7}{2}}^{(i+2)} + 9b_{\frac{7}{2}}^{(i)} + 3b_{\frac{7}{2}}^{(i-2)}, \\
\mathsf{B}_{\frac{7}{2}}^{(i,2)} &= 3b_{\frac{7}{2}}^{(i+1)} + 3b_{\frac{7}{2}}^{(i-1)}, \\
\mathsf{B}_{\frac{7}{2}}^{(i,3)} &= b_{\frac{7}{2}}^{(i)}.
\end{aligned}$$

18. In computing the factors of the preceding formula which depend on E and E', the following abbreviation can be used. M_n denoting the factor which multiplies

$$\pm \frac{1}{n!} a^{k+n} \frac{d^n \mathsf{B}_{\frac{2k+1}{2}}}{da^n},$$

and Δ being the symbol of finite differences with respect to n, it is plain that

$$\Delta^n M_n = (-1)^n \mathsf{E}_{k+n} \mathsf{E}'_{-(k+n+1)}.$$

Hence, if the products $\mathsf{E}_{k+n}\,\mathsf{E}'_{-(k+n+1)}$ are computed for the various values of n, and are taken alternately with the positive and negative sign, and are written as if they were the successive differences of a function, we shall get the values of the factors M_n by filling out the scheme of differences. This abbreviation is applicable equally whether we are making a numerical

computation of the coefficient or a literal one. In the latter case the abbreviation can be applied separately to each term of the form $Ce^{i}e'^{i'}$ in the products $\mathsf{E}_k \mathsf{E}'_{-(k+1)}$.

19. We proceed now to discuss the functions E. From their definition we have

$$\left(\frac{r}{a}\right)^{i} \varepsilon^{jv\sqrt{-1}} = \sum_{k=-\infty}^{k=+\infty} \mathsf{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} z^{k},$$

whence

$$\left(\frac{r}{a}\right)^{i} \cos jv = \tfrac{1}{2} \sum_{k=-\infty}^{k=+\infty} \left[\mathsf{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} + \mathsf{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix} \right] \cos k\zeta,$$

$$\left(\frac{r}{a}\right)^{i} \sin jv = \tfrac{1}{2} \sum_{k=-\infty}^{k=+\infty} \left[\mathsf{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} - \mathsf{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix} \right] \sin k\zeta.$$

From which we gather that the functions E can be computed by definite integrals, thus

$$\mathsf{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \frac{1}{\pi} \int_0^{\pi} \left(\frac{r}{a}\right)^{i} \cos(jv - k\zeta)\, d\zeta.$$

Let us now suppose that the coefficient of s^k, in the expansion of

$$\eta^{i} s^{j} (1 - \omega s)^{i-j} \left(1 - \frac{\omega}{s}\right)^{i+j}$$

in powers of s, is denoted by $E\begin{pmatrix} i \\ j \\ k \end{pmatrix}$, then evidently

$$\mathsf{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \sum_{l=-\infty}^{l=+\infty} \cdot E\begin{pmatrix} i+1 \\ j \\ k-l \end{pmatrix} J^{(l)}_{\frac{ke}{2}}.$$

By writing in the expression $1 \div s$ for s and changing the sign of j, it remains unaltered; hence the relation

$$E\begin{pmatrix} i \\ -j \\ -k \end{pmatrix} = E\begin{pmatrix} i \\ j \\ k \end{pmatrix}.$$

By developing the factors of the expression

$$\eta^{i} s^{j} (1 - \omega s)^{i-j} \left(1 - \frac{\omega}{s}\right)^{i+j}$$

by the binomial theorem, we get

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j} [i-j,\, k-j]\, \eta^{i} \omega^{k-j}$$
$$\times \left[1 + \frac{(i+j)(i-k)}{1.(k-j+1)} \omega^{2} + \frac{(i+j)(i+j-1)(i-k)(i-k-1)}{1.2.(k-j+1)(k-j+2)} \omega^{4} + \ldots \right].$$

This equation, as written, is correct only when $k-j$ is not negative, but by the relation given above we can reduce the case of $k-j$ negative to that where it is positive. The factor in the brackets is a case of the series

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma\,.\,(\gamma+1)}x^2+\ldots$$

treated by Gauss in a memoir entitled "*Disquisitiones generales circa seriem infinitam, &c.*" (See Gauss' Werke, Vol. III, p. 123, and especially the "Nachlass," p. 207.) According to Gauss' notation

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix}=(-1)^{k-j}[i-j,\,k-j]\,\eta^i\omega^{k-j}F(-i-j,\,k-i,\,k-j+1,\,\omega^2).$$

Whenever, of $i+j$ and $i-j$, one is not negative, this series terminates after a certain number of terms, thus affording a finite expression for the function. But when these integers are both negative, the series is infinite. However, it can be easily transformed into another which like the former is finite. From Gauss' investigation of these series (see the volume just quoted, p. 209, equation [82]), we have

$$F(\alpha,\beta,\gamma,x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma,x).$$

Applying this to our expression, we get

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix}=(-1)^{k-j}[i-j,\,k-j]\,\eta^i\omega^{k-j}(1-\omega^2)^{2i+1}F(i+k+1,\,i-j+1,\,k-j+1,\,\omega^2).$$

This expression is evidently finite when $i-j$ and $i+j$ are negative.

20. The developments of the functions E in powers of e as far as e^7 have been tabulated by Prof. Cayley in the Memoirs of the Royal Astronomical Society, Vol. XXVII. It would conduce to the ready employment of the preceding formulas if we had the function $\mathrm{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix}$ explicitly expanded in ascending powers of e, but the attempts I have made to write such a series lead to extremely complex forms of the coefficients. Hence I shall give here only the coefficients of the lowest power of e in this function, which suffices for obtaining all the terms of the lowest order in any coefficient of the expansion of $1\div\Delta$. We have, when $j-k$ is positive,

$$\mathrm{E}\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix}=\Big[[i+j,j-k]+[i+j,j-k-1]\frac{k}{1}+[i+j,j-k-2]\frac{k^2}{1.2}+\ldots.+[i+j,0]\frac{k^{j-k}}{(j-k)!}\Big]\left(-\frac{e}{2}\right)^{j-k},$$

and when $k-j$ is positive

$$\mathrm{E}\binom{i-1}{\substack{j\\k}} = \left[[i-j,\, k-j] - [i-j,\, k-j-1]\frac{k}{1} + [i-j,\, k-j-2]\frac{k^2}{1.2} \right.$$
$$\left. - \ldots . + [i-j,\, 0]\frac{(-k)^{k-j}}{(k-j)!}\right]\left(-\frac{e}{2}\right)^{k-j}.$$

21. Thus in the example alluded to above, of the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$, we find that the terms of the lowest order in E and E' (omitting here, as in the scheme, the two lower indices), are

$$\mathrm{E}_0 = \mathrm{E}_1 = \mathrm{E}_2 = \mathrm{E}_3 = 1,$$

$$\mathrm{E}'_{-1} = -\left[[-2, 3] - [-2, 2]\tfrac{5}{1} + [-2, 1]\frac{5^2}{1.2} - [-2, 0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \tfrac{389}{48}e'^3.$$

$$\mathrm{E}'_{-2} = -\left[[-3, 3] - [-3, 2]\tfrac{5}{1} + [-3, 1]\frac{5^2}{1.2} - [-3, 0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \tfrac{590}{48}e'^3,$$

$$\mathrm{E}'_{-3} = -\left[[-4, 3] - [-4, 2]\tfrac{5}{1} + [-4, 1]\frac{5^2}{1.2} - [-4, 0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \tfrac{845}{48}e'^3,$$

$$\mathrm{E}'_{-4} = -\left[[-5, 3] - [-5, 2]\tfrac{5}{1} + [-5, 1]\frac{5^2}{1.2} - [-5, 0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \tfrac{1160}{48}e'^3.$$

Bringing into use our method of abbreviation, we multiply each of the preceding numerical coefficients by 48 in order to avoid fractions, and then write them alternately with the positive and negative signs in a diagonal line, and from these, as successive orders of differences, derive the numbers standing in the vertical columns, thus:

$$\begin{array}{rrrr}
+389 & & & \\
 & -590 & & \\
-201 & & +845 & \\
 & +255 & & -1160 \\
+54 & & -315 & \\
 & -60 & & +381 \\
-6 & & +66 & \\
 & +6 & & -72 \\
0 & & -6 & \\
 & 0 & & +6 \\
 & & 0 & \\
 & & & 0
\end{array}$$

and dividing the numbers of the first column respectively by $1,\ -1,\ 1.2$ $-1.2.3$, we get the following as the terms of the lowest order in the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$ in $a' \div \Delta$,

$$\frac{1}{48}\left[389 b_{\frac{1}{2}}^{(2)} + 201\, a\frac{db_{\frac{1}{2}}^{(2)}}{da} + 27 a^2\frac{d^2 b_{\frac{1}{2}}^{(2)}}{da^2} + a^3\frac{d^3 b_{\frac{1}{2}}^{(2)}}{da^3}\right]e'^3,$$

which agrees with that found in the books. The following additional terms

of the same coefficient can be written from the second, third, &c., columns, viz., those which are multiplied by e'^3 and the various powers of $\sin^2 \frac{1}{2} I$,

$$-\frac{1}{2}\frac{1}{48}\left[590 a\, B_{\frac{3}{2}}^{(2,0)} + 255 a^2 \frac{dB_{\frac{3}{2}}^{(2,0)}}{da} + 30 a^3 \frac{d^2 B_{\frac{3}{2}}^{(2,0)}}{da^2} + a^4 \frac{d^3 B_{\frac{3}{2}}^{(2,0)}}{da^3}\right] e'^3 \sin^2 \frac{I}{2}$$
$$+\frac{1.3}{2.4}\frac{1}{48}\left[845 a^2 B_{\frac{5}{2}}^{(2,0)} + 315 a^3 \frac{dB_{\frac{5}{2}}^{(2,0)}}{da} + 33 a^4 \frac{d^2 B_{\frac{5}{2}}^{(2,0)}}{da^2} + a^5 \frac{d^3 B_{\frac{5}{2}}^{(2,0)}}{da^3}\right] e'^3 \sin^4 \frac{I}{2}$$
$$- \&c. \; . \; . \; . \; . \; .$$

22. When we wish to obtain only the terms independent of ζ and ζ', that is, those on which the secular perturbations depend, $i = 0$ and $i' = 0$, and the Besselian function J disappears from the expressions giving the values of E and E′, and the coefficient of $\cos (i''\gamma + i'''\theta)$ in the expansion of $\frac{1}{\Delta}$ can be written

$$\frac{1}{a'} \sum_{k=i'''}^{k=\infty} \sum_{n=0}^{n=\infty} \sum_{p=0}^{p=n} (-1)^{k-i'''+n-p} \frac{[k][n,p]}{n!}$$
$$\times a^{k+n} \frac{d^n B_{\frac{2k+1}{2}}^{(i'',i''')}}{da^n} \sin^{2k} \frac{I}{2} \cdot E\begin{pmatrix} k+p+1 \\ i''+i''' \\ 0 \end{pmatrix} \cdot E'\begin{pmatrix} -((k+p) \\ -i'''-i'') \\ 0 \end{pmatrix}.$$

23. In leaving the subject of the development of $1 \div \Delta$, it may be well to note that two other forms can be given to the expression of the general coefficient, by employing, instead of the expression given above, either of the following:

$$\frac{r}{r'} - a = a \frac{\frac{e'}{2}\left(s' + \frac{1}{s'}\right) - \frac{e}{2}\left(s + \frac{1}{s}\right)}{1 - \frac{e'}{2}\left(s' + \frac{1}{s'}\right)}$$
$$= a \frac{\frac{e'}{2}\left(s' + \frac{1}{s'}\right) - \frac{e}{2}\left(s + \frac{1}{s}\right)}{\eta' (1 - \omega' s')\left(1 - \frac{\omega'}{s'}\right)}.$$

But as they do not possess as much symmetry and brevity as the form given above, we will pass over them.

24. The second part of the Perturbative Function, omitting the factor m', is

$$-\frac{r}{r'^2}\cos\psi = -\frac{r}{r'^2}\left[\cos^2\frac{I}{2}\cos(v - v' + \gamma) + \sin^2\frac{I}{2}\cos(v + v' + \theta)\right]$$
$$= -\tfrac{1}{2}\frac{r}{r'^2}\cos^2\frac{I}{2}\left[g\varepsilon^{(v-v')\sqrt{-1}} + g^{-1}\varepsilon^{-(v-v')\sqrt{-1}}\right]$$
$$-\tfrac{1}{2}\frac{r}{r'^2}\sin^2\frac{I}{2}\left[h\varepsilon^{(v+v')\sqrt{-1}} + h^{-1}\varepsilon^{-(v+v')\sqrt{-1}}\right].$$

According to the first theorem of §3, the coefficient of z^0 in $\frac{r}{a}\,\varepsilon^{v\sqrt{-1}}$ is equal to that of s^0 in

$$\eta^2 s\,(1-\omega s)\left(1-\frac{\omega}{s}\right)^3;$$

or it is equal to

$$-3\eta^2\omega\,(1+\omega^2)=-\tfrac{3}{2}e.$$

And, according to the second theorem, the coefficient of z^i in the same function is equal to that of s^i in

$$\frac{\eta}{i}\frac{d}{ds}\left[s\left(1-\frac{\omega}{s}\right)^2\right].\,s\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}=\frac{\eta}{i}\left(s-\frac{\omega^2}{s}\right)\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}.$$

Hence we have

$$\frac{r}{a}\,\varepsilon^{v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}\frac{\eta}{i}\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

And by simply writing $1\div z$ for z,

$$\frac{r}{a}\,\varepsilon^{-v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}\frac{\eta}{i}\left[\omega^2J_{\frac{ie}{2}}^{(i-1)}-J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

The well-known differential equations of elliptic motion

$$\frac{d^2x}{d\zeta^2}+\frac{a^3}{r^3}x=0,$$
$$\frac{d^2y}{d\zeta^2}+\frac{a^3}{r^3}y=0,$$

supposing the axis of x to be directed towards the perihelion, give us the equation

$$\frac{a^3}{r^3}\,\varepsilon^{v\sqrt{-1}}=-\frac{d^2\left(\frac{r}{a}\,\varepsilon^{v\sqrt{-1}}\right)}{d\zeta^2},$$

and consequently these two

$$\frac{a^3}{r^3}\,\varepsilon^{v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}i\eta\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

$$\frac{a^3}{r^3}\,\varepsilon^{-v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}i\eta\left[\omega^2J_{\frac{ie}{2}}^{(i-1)}-J_{\frac{ie}{2}}^{(i+1)}\right]z^i.$$

By substituting these values in the expression given above for $-\frac{r}{r'^2}\cos\psi$, it is not difficult to see that, in it, the coefficient of

$$\cos(i\zeta+i'\zeta'+\gamma)$$

is

$$-\frac{a}{a'^2}\cos^2\frac{I}{2}\,.\,\frac{\eta}{i}\left[J_{\frac{ie}{2}}^{(i-1)}-\omega^2J_{\frac{ie}{2}}^{(i+1)}\right].\,i'\eta'\left[\omega'^2J_{\frac{i'e'}{2}}^{(i'-1)}-J_{\frac{i'e'}{2}}^{(i'+1)}\right],$$

and the coefficient of

$$\cos(i\zeta + i'\zeta' + \theta)$$

is

$$-\frac{a}{a'^2}\sin^2\frac{I}{2}\cdot\frac{\eta}{i}\left[J^{(i-1)}_{\frac{ie}{2}} - \omega^2 J^{(i+1)}_{\frac{ie}{2}}\right]\cdot i'\eta'\left[J^{(i'-1)}_{\frac{i'e'}{2}} - \omega'^2 J^{(i'+1)}_{\frac{i'e'}{2}}\right].$$

In the special case of $i = 0$ the middle factors of these expressions take the indeterminate form $0 \div 0$, but then, in accordance with what has been shown above we should read $-\frac{3}{2}e$. Thus, by means of the Besselian functions, these coefficients take finite forms.

MEMOIR No. 26

Demonstration of the Differential Equations Employed by Delaunay in the Lunar Theory.

(The Analyst, Vol. III, pp. 65–70, 1876.)

The method of treating the lunar theory adopted by Delaunay is so elegant that it cannot fail to become in the future the classic method of treating all the problems of celestial mechanics. The canonical system of equations employed by Delaunay is not demonstrated by him in his work, but he refers to a memoir of Binet inserted in the *Journal de l'École Polytechnique*, Cahier XXVIII. Among the innumerable sets of canonical elements it does not appear that a better can be selected. These equations can be established in a very elegant manner by using the properties of Lagrange's and Poisson's quantities (a, b) and $[a, b]$. But a demonstration founded on more direct and elementary considerations, is, on some accounts, to be preferred.

Let a denote the mean distance, e the eccentricity, i the inclination of the orbit to a fixed plane, l the mean anomaly, g the angular distance of the lower apsis from the ascending node, h the longitude of the ascending node measured from a fixed line in the fixed plane, μ the sum of the masses of the bodies whose relative motion is considered, and R the ordinary perturbative function augmented by the term $\frac{\mu^2}{2L^2}$. Then if we put $L = \sqrt{\mu a}$, $G = \sqrt{[\mu a(1-e^2)]}$, $H = \sqrt{[\mu a(1-e^2)]}\cos i$, Delaunay's equations are

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R}{\partial h},$$
$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H},$$

In terms of rectangular coordinates

$$R = \frac{\mu^2}{2L^2} + \frac{m'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{\frac{1}{2}}} - \frac{m'(xx' + yy' + zz')}{r'^3}.$$

In this expression, for x, y, z, ought to be substituted their values deduced from the formulas of elliptic motion, and expressed in terms of L, G, H, l, g, h. It should be noted that the term $\frac{\mu^2}{2L^2} = \frac{\mu}{2a}$, of the zero order with respect to the disturbing force, has been added to R only to preserve

in the equations the canonical form: it is only by amplifying the signification of the word that l can be called an element, as it is not constant in elliptic motion, but augments proportionally to the time and $\frac{dl}{dt} = n = \frac{\mu^2}{L^3}$. It is chosen as a variable in preference to the element attached to it by addition simply to prevent t from appearing in derivatives of R outside of the functional signs sine and cosine.

The equations

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = \frac{\partial R}{\partial x}, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} = \frac{\partial R}{\partial y}, \quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} = \frac{\partial R}{\partial z},$$

are well known; here, however, R does not contain the term $\frac{\mu^2}{2L^2}$. By multiplying them severally by dx, dy, dz, adding and integrating, is obtained

$$\frac{dx^2 + dy^2 + dz^2}{2dt^2} - \frac{\mu}{r} + \frac{\mu}{2a} = \int \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz \right).$$

When the elements are made variable, this gives

$$\frac{d}{dt}\left(\frac{\mu}{2a}\right) = -\left(\frac{\partial R}{\partial x}\frac{dx}{dt} + \frac{\partial R}{\partial y}\frac{dy}{dt} + \frac{\partial R}{\partial z}\frac{dz}{dt}\right).$$

But we have

$$\frac{dx}{dt} = n\frac{dx}{dl}, \quad \frac{dy}{dt} = n\frac{dy}{dl}, \quad \frac{dz}{dt} = n\frac{dz}{dl},$$

and hence

$$\frac{d}{dt}\left(\frac{\mu}{2a}\right) = -n\left(\frac{\partial R}{\partial x}\frac{\partial x}{\partial l} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial l} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial l}\right) = -n\frac{\partial R}{\partial l}.$$

Dividing both members of this equation by $-n = -\sqrt{\mu a^{-3}}$, the left member is seen to be the differential of $\sqrt{\mu a} = L$. Consequently,

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}.$$

Denoting the true anomaly by v, the orthogonal projection of the radius vector on the line of nodes is $r\cos(v+g)$, and on a line perpendicular to it and in the plane of the orbit $r\sin(v+g)$. And the latter projected on the plane of reference is $r\sin(v+g)\cos i$, and on a line perpendicular to this plane $r\sin(v+g)\sin i$. If the two projections lying in the plane of reference are again each projected on the axis of x, their sum will be the value of the coordinate x, and the sum of their projections on the axis of y, the value of the coordinate y. Hence

$$x = r\cos(v+g)\cos h - r\sin(v+g)\cos i \sin h,$$
$$y = r\cos(v+g)\sin h + r\sin(v+g)\cos i \cos h,$$
$$z = r\sin(v+g)\sin i,$$

or, substituting for i its value in terms of G and H,

$$x = r\cos(v+g)\cos h - \frac{H}{G} r \sin(v+g)\sin h,$$

$$y = r\cos(v+g)\sin h + \frac{H}{G} r \sin(v+g)\cos h,$$

$$z = \frac{\sqrt{G^2 - H^2}}{G} r \sin(v+g).$$

As r and v are functions of L, G and l only, the preceding equations show the manner in which H, g and h are involved in R.

H denotes double the areal velocity projected on the plane xy, or

$$H = \frac{xdy - ydx}{dt}.$$

Consequently

$$\frac{dH}{dt} = x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x}.$$

But the foregoing values of x, y, z show that we have

$$\frac{\partial x}{\partial h} = -y, \quad \frac{\partial y}{\partial h} = x, \quad \frac{\partial z}{\partial h} = 0;$$

and thus

$$\frac{dH}{dt} = \frac{\partial R}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial h} = \frac{\partial R}{\partial h}.$$

G denotes double the areal velocity, and evidently, if for the moment we suppose x and y to be drawn in the plane of the orbit, the axis of x towards the lower apsis,

$$\frac{dG}{dt} = x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x} = \frac{\partial R}{\partial v},$$

where, in the last R, for x, y, z must be substituted their values given above in terms of r, v, G, H, g, h. Now, as the only way in which g is involved in these values, is by addition to v, it follows that

$$\frac{\partial R}{\partial v} = \frac{\partial R}{\partial g};$$

and this equation is not affected when, for r and v in R, are substituted their values in terms of L, G and l. Consequently

$$\frac{dG}{dt} = \frac{\partial R}{\partial g}.$$

In the elliptic theory

$$\frac{xdz - zdx}{dt} = \sqrt{G^2 - H^2}\cos h,$$

$$\frac{ydz - zdy}{dt} = \sqrt{G^2 - H^2}\sin h.$$

Whence we deduce

$$\frac{d\left[\sqrt{G^2 - H^2}\cos h\right]}{dt} = x\frac{\partial R}{\partial z} = z\frac{\partial R}{\partial x},$$

$$\frac{d\left[\sqrt{G^2 - H^2}\sin h\right]}{dt} = y\frac{\partial R}{\partial z} = z\frac{\partial R}{\partial y}.$$

Eliminating $d\sqrt{G^2 - H^2}$ from these equations, we obtain

$$\frac{dh}{dt} = \frac{z\sin h}{\sqrt{G^2 - H^2}}\frac{\partial R}{\partial x} - \frac{z\cos h}{\sqrt{G^2 - H^2}}\frac{\partial R}{\partial y} - \frac{x\sin h - y\cos h}{\sqrt{G^2 - H^2}}\frac{\partial R}{\partial z}.$$

Comparing the coefficients of the three derivatives of R in the right member of this equation with the values of x, y and z in terms of r, v, G, H, g, h, we recognize that they are severally equivalent to the negatives of the partial derivatives of these quantities with respect to H. So that

$$\frac{dh}{dt} = -\left(\frac{\partial R}{\partial x}\frac{\partial x}{\partial H} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial H} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial H}\right) = -\frac{\partial R}{\partial H}.$$

It is a well-known principle in the theory of varying elements, that if we differentiate any function, which is a function of the coordinates and t only, but expressed in terms of t and the elements, with respect to t only inasmuch as it is explicitly involved, we obtain the correct value. Hence, if the differentiation is performed on the supposition that the elements are alone variable, the result should be zero. Applying this to the function r we get

$$\frac{\partial r}{\partial L}\frac{dL}{dt} + \frac{\partial r}{\partial G}\frac{dG}{dt} + \frac{\partial r}{\partial l}\left(\frac{dl}{dt} - n\right) = 0,$$

or

$$\frac{\partial r}{\partial L}\frac{\partial R}{\partial l} + \frac{\partial r}{\partial G}\frac{\partial R}{\partial g} + \frac{\partial r}{\partial l}\left(\frac{dl}{dt} - n\right) = 0,$$

or again

$$\frac{\partial r}{\partial L}\left(\frac{\partial R}{\partial r}\frac{\partial r}{\partial l} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial l}\right) + \frac{\partial r}{\partial G}\frac{\partial R}{\partial v} + \frac{\partial r}{\partial l}\left(\frac{dl}{dt} - n\right) = 0.$$

Whence we derive

$$\frac{dl}{dt} = n - \frac{\partial r}{\partial L}\frac{\partial R}{\partial r} - \left(\frac{\partial r}{\partial l}\right)^{-1}\left[\frac{\partial r}{\partial L}\frac{\partial v}{\partial l} + \frac{\partial r}{\partial G}\right]\frac{\partial R}{\partial v}.$$

From the expression for r we can eliminate l and introduce v in its place by means of the expression for v in terms of L, G and l; the result is the well-known equation

$$r = \frac{a(1 - e^2)}{1 + e\cos v} = \frac{G^2}{\mu\left[1 + \frac{\sqrt{L^2 - G^2}}{L}\cos v\right]}.$$

And we have

$$\frac{\partial r}{\partial l}=\frac{\partial r}{\partial v}\frac{\partial v}{\partial l}, \quad \frac{\partial r}{\partial L}=\left(\frac{\partial r}{\partial L}\right)+\frac{\partial r}{\partial v}\frac{\partial v}{\partial L},$$

the parentheses denoting the derivative with respect to L only insomuch as it enters the preceding equation for r. By making these substitutions, the coefficient of $\frac{\partial R}{\partial v}$ in the expression for $\frac{dl}{dt}$ becomes

$$-\frac{\partial v}{\partial L}-\left(\frac{\partial r}{\partial l}\right)^{-1}\left[\left(\frac{\partial r}{\partial L}\right)\frac{\partial v}{\partial l}+\frac{\partial r}{\partial G}\right].$$

From the preceding equation for r, we derive

$$\left(\frac{\partial r}{\partial L}\right)=-\frac{\mu r^2 \cos v}{L^3 e},$$

also the following is a well-known equation in the elliptic theory

$$\frac{\partial v}{\partial l}=\frac{G}{nr^2}.$$

For obtaining the value of $\frac{\partial r}{\partial G}$, u being the eccentric anomaly, we have the equations

$$r=a(1-e\cos u), \quad l=u-e\sin u.$$

Their differentials give

$$\frac{\partial r}{\partial e}=-a\cos u+ae\sin u\frac{\partial u}{\partial e},$$

$$0=(1-e\cos u)\,du-\sin u\,.\,de.$$

Whence

$$\frac{\partial r}{\partial e}=-a\frac{\cos u-e}{1-e\cos u}=-a\cos v.$$

And

$$e=\frac{\sqrt{L^2-G^2}}{L}, \quad \frac{\partial e}{\partial G}=-\frac{G}{L^2 e},$$

$$\frac{\partial r}{\partial G}=\frac{\partial r}{\partial e}\frac{\partial e}{\partial G}=\frac{G\cos v}{\mu e}.$$

By substituting the values, it is found that

$$\left(\frac{\partial r}{\partial L}\right)\frac{\partial v}{\partial l}+\frac{\partial r}{\partial G}=0.$$

In consequence

$$\frac{dl}{dt}=n-\frac{\partial R}{\partial r}\frac{\partial r}{\partial L}-\frac{\partial R}{\partial v}\frac{\partial v}{\partial L}=n-\frac{\partial R}{\partial L}.$$

As R is a function of the coordinates and the time only, we can treat it as we have done r. Then

$$\frac{\partial R}{\partial L}\frac{dL}{dt}+\frac{\partial R}{\partial l}\left(\frac{dl}{dt}-n\right)+\frac{\partial R}{\partial G}\frac{dG}{dt}+\frac{\partial R}{\partial g}\frac{dg}{dt}+\frac{\partial R}{\partial H}\frac{dH}{dt}+\frac{\partial R}{\partial h}\frac{dh}{dt}=0.$$

On substituting in this the values of the differentials of the elements which have already been determined, it is seen that all the terms but two, mutually cancel each other. And, on dividing the result by $\frac{\partial R}{\partial g}$, we get

$$\frac{dg}{dt}=-\frac{\partial R}{\partial G}.$$

By adding to R the term $\frac{\mu^2}{2L^2}=\frac{\mu}{2a}$, its partial derivative with respect to L is augmented by the term $-\frac{\mu^2}{L^3}=-n$, but all the other derivatives are unchanged. In consequence of this addition, the value of the differential of l becomes

$$\frac{dl}{dt}=-\frac{\partial R}{\partial L}.$$

An objection may be made against the preceding method of obtaining the differentials of l and g, that the quantities $\frac{\partial r}{\partial l}$ and $\frac{\partial R}{\partial g}$, which both periodically vanish, have been employed as divisors. But this objection has force only when it is admitted that the differentials of l and g, or the corresponding derivatives of R, may be discontinuous. For, having proved the truth of the equation for all times, except when the divisors, just mentioned, vanish, it follows, that if both members are continuous, the equations must still hold even for the moments of time when $\frac{\partial r}{\partial l}=0$ or $\frac{\partial R}{\partial g}=0$.

MEMOIR No. 27.

Solution of a Problem in the Motion of Rolling Spheres.

(The Analyst, Vol. III, pp. 92-93, 1876.)

A sphere, of radius r, rolls down the surface of another sphere, of the same material, of radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. It is proposed to determine the motion of the sphere, the point of separation, and the equation of the curve described by the centre of the upper sphere.

Let x and 0 be the coordinates of the center of the lower sphere, x' and y' those of the center of the upper, θ and θ' the amounts of rotation, and ϕ the angle the line joining their centers makes with the horizon, and for brevity put $h = R + r$.

The expression for the living force is

$$T = \frac{m}{2}\left[\frac{dx^2}{dt^2} + \tfrac{2}{5}R^2\frac{d\theta^2}{dt^2}\right] + \frac{m'}{2}\left[\frac{dx'^2}{dt^2} + \frac{dy'^2}{dt^2} + \tfrac{2}{5}r^2\frac{d\theta'^2}{dt^2}\right],$$

and the potential is $\Omega = - m'gy'$.

According to the frictional conditions, the variables x, x', y', θ and θ' satisfy the following equations:

$$\left.\begin{aligned} &R\theta - x &&= 0, \\ &r\theta' + x + h\tan^{-1}\frac{y'}{x'-x} &&= 0, \\ &\sqrt{[(x'-x)^2 + y'^2]} - h &&= 0.\text{*} \end{aligned}\right\} \tag{1}$$

With Lagrange's method of multipliers, if we denote these equations respectively by $L = 0$, $M = 0$, $N = 0$, and the multipliers of their differentials by λ, μ, ν, and take ξ to represent any one of the five variables x, x', y', θ, θ', the general equation of the problem is

$$\frac{d}{dt}\frac{\partial T}{\partial\frac{d\xi}{dt}} - \frac{\partial T}{\partial\xi} = \frac{\partial\Omega}{\partial\xi} + \lambda\frac{\partial L}{\partial\xi} + \mu\frac{\partial M}{\partial\xi} + \nu\frac{\partial N}{\partial\xi}.$$

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* These eqnations snbsist only as long as the spheres are in contact.

Applying this in succession to each of the five variables, and writing for simplicity ϕ for $\tan^{-1}\dfrac{y'}{x'-x}$, we get

$$\left.\begin{aligned}
m\frac{d^2x}{dt^2} &= -\lambda + \mu(1+\sin\varphi) - \nu\cos\varphi\,,\\
m'\frac{d^2x'}{dt^2} &= -\mu\sin\varphi + \nu\cos\varphi\,,\\
m'\frac{d^2y'}{dt^2} &= -m'g + \mu\cos\varphi + \nu\sin\varphi\,,\\
\tfrac{2}{5}mR^2\frac{d^2\theta}{dt^2} &= \lambda R\,,\\
\tfrac{2}{5}m'r^2\frac{d^2\theta'}{dt^2} &= \mu r\,.
\end{aligned}\right\} \tag{2}$$

Adding the first and second of (2),

$$\frac{d^2(mx+m'x')}{dt^2} = \mu - \lambda\,.$$

The two first of (1) and the two last of (2) give

$$\begin{aligned}
\lambda &= \tfrac{2}{5}mR\frac{d^2\theta}{dt^2} = \tfrac{2}{5}m\frac{d^2x}{dt^2}\,,\\
\mu &= \tfrac{2}{5}m'r\frac{d^2\theta'}{dt^2} = -\tfrac{2}{5}m'\left[\frac{d^2x}{dt^2} + h\frac{d^2\varphi}{dt^2}\right].
\end{aligned}$$

Substituting these values for λ and μ in the last equation,

$$\frac{d^2(mx+m'x')}{dt^2} = -\tfrac{2}{5}(m+m')\frac{d^2x}{dt^2} - \tfrac{2}{5}m'h\frac{d^2\varphi}{dt^2}\,. \tag{3}$$

Integrating once and eliminating x',

$$\tfrac{7}{5}(m+m')\frac{dx}{dt} + m'h\left(\tfrac{2}{5} - \sin\varphi\right)\frac{d\varphi}{dt} = 0\,,$$

where the constant is zero because the spheres are supposed to set out together from a state of rest. As $\dfrac{d\phi}{dt}$, in general, is negative (ϕ can always be supposed in the first quadrant), it is evident from this equation, that if $\sin\phi > \frac{2}{5}$, the lower sphere will move horizontally towards the side on which the upper sphere is; but if $\sin\phi < \frac{2}{5}$, in the opposite direction.

Integrating (3) twice

$$(7m+2m')x + 5m'x' + 2m'h\varphi = a \text{ constant}.$$

Eliminating x and ϕ from this by substituting their values in terms of x' and

y', we get as the equation of the path of the center of the upper sphere

$$7(m+m')[x'-\sqrt{h^2-y'^2}]+m'\left[2h\sin^{-1}\frac{y'}{h}+5\sqrt{h^2-y'^2}\right]=\text{a constant.}$$

As ν denotes the pressure of the upper on the lower sphere, the spheres will separate when $\nu=0$. Now if we eliminate μ between the second and third of (2), we see that $\nu=0$ is equivalent to

$$\frac{d^2x'}{dt^2}\cos\varphi+\left(\frac{d^2y'}{dt^2}+g\right)\sin\varphi=0.$$

And if we eliminate x' and y' from this by means of their values in terms of x and ϕ, we get

$$\frac{d^2x}{dt^2}\cos\varphi+g\sin\varphi-h\frac{d\varphi^2}{dt^2}=0.$$

By eliminating second derivatives this becomes

$$49m\left[\frac{h}{g}\frac{d\varphi^2}{dt^2}-\sin\varphi\right]+10m'(1+\sin\varphi)^2\left[\frac{h}{g}\frac{d\varphi^2}{dt^2}-1\right]=0,$$

which, by substituting the value of $\frac{d\phi}{dt}$, becomes (β is the initial value of ϕ)

$$\begin{aligned}&70(m+m')[49m+10m'+20m'\sin\varphi+10m'\sin^2\varphi][\sin\beta-\sin\varphi]\\&-[10m'+(49m+20m')\sin\varphi+10m'\sin^2\varphi][49m+45m'+20m'\sin\varphi-25m'\sin^2\varphi]=0.\end{aligned}$$

MEMOIR No. 28.

Reduction of the Problem of Three Bodies.

(The Analyst, Vol. III, pp. 179-185, 1876.)

The object of this article is to find the three differential equations which virtually determine the sides of the triangle formed by the three bodies, bringing to our aid all the known finite integrals of the problem.

Lagrange was the first to treat this question in his *Essai sur le Problème des Trois Corps* (Oeuvres, Tome VI, p. 227); but the formulas lacking symmetry, his editor, Serret, has, in a note, supplied this and pointed out an important error into which Otto Hesse, who had investigated this subject (*Journal für die Mathematik*, Band LXXIV) had fallen.

By adopting an orthogonal substitution, at the outset, for reducing the number of coordinates from nine to six, we can prevent the masses from entering the equations except through the potential function or its derivatives. In this way symmetry, indeed, appears to be lost, but there is so great a gain in condensation of the formulas, that we can carry out some of the eliminations which previous writers have been content only to indicate.

Let ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ'' be the rectangular coordinates of the masses m, m', m'', the expression for the living force will be

$$T = m\frac{d\xi^2 + d\eta^2 + d\zeta^2}{2dt^2} + m'\frac{d\xi'^2 + d\eta'^2 + d\zeta'^2}{2dt^2} + m''\frac{d\xi''^2 + d\eta''^2 + d\zeta''^2}{2dt^2};$$

and Δ, Δ', Δ'' being given by the equations

$$\begin{aligned}
\Delta^2 &= (\xi' - \xi'')^2 + (\eta' - \eta'')^2 + (\zeta' - \zeta'')^2, \\
\Delta'^2 &= (\xi'' - \xi)^2 + (\eta'' - \eta)^2 + (\zeta'' - \zeta)^2, \\
\Delta''^2 &= (\xi - \xi')^2 + (\eta - \eta')^2 + (\zeta - \zeta')^2,
\end{aligned}$$

the potential function

$$\Omega = \frac{m'm''}{\Delta} + \frac{mm''}{\Delta'} + \frac{mm'}{\Delta''}.$$

Without lessening the generality, the origin of coordinates can be put

at the center of gravity, when the principle of the conservation of this center will furnish the equations

$$\left.\begin{aligned} m\xi + m'\xi' + m''\xi'' &= 0, \\ m\eta + m'\eta' + m''\eta'' &= 0, \\ m\zeta + m'\zeta' + m''\zeta'' &= 0, \end{aligned}\right\} \tag{1}$$

By means of these relations three of the variables can be eliminated and the number thus reduced from nine to six. This transformation is most elegantly accomplished by putting

$$\begin{aligned} \xi &= \alpha x + \beta x', & \eta &= \alpha y + \beta y', & \zeta &= \alpha z + \beta z', \\ \xi' &= \alpha' x + \beta' x', & \eta' &= \alpha' y + \beta' y', & \zeta' &= \alpha' z + \beta' z', \\ \xi'' &= \alpha'' x + \beta'' x', & \eta'' &= \alpha'' y + \beta'' y', & \zeta'' &= \alpha'' z + \beta'' z', \end{aligned}$$

where α, α', α'', β, β', β'' are six constants which may be so taken that they satisfy the five equations

$$\left.\begin{aligned} m\alpha + m'\alpha' + m''\alpha'' &= 0, \\ m\beta + m'\beta' + m''\beta'' &= 0, \\ m\alpha\beta + m'\alpha'\beta' + m''\alpha''\beta'' &= 0, \\ m\alpha^2 + m'\alpha'^2 + m''\alpha''^2 &= 1, \\ m\beta^2 + m'\beta'^2 + m''\beta''^2 &= 1. \end{aligned}\right\} \tag{2}$$

The first two are necessary in order that equations (1) may be satisfied; the third is adopted in order that nothing but squares of differential coefficients may occur in the transformed T; and, evidently, the last two may be adopted without thereby diminishing the generality of the transformation.

These equations may be solved elegantly in the following manner: Put

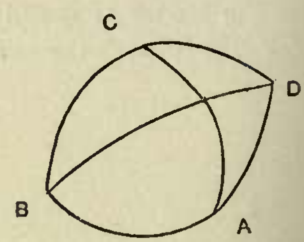
$$\sqrt{m} = k \sin\gamma \cos\varepsilon, \quad \sqrt{m'} = k \sin\gamma \sin\varepsilon, \quad \sqrt{m''} = k \cos\gamma;$$

and adopt the four quantities ϕ, ϕ', ω, ω', such that

$$\begin{aligned} \sqrt{m}\,\alpha &= \sin\varphi \cos(\omega + \varepsilon), & \sqrt{m}\,\beta &= \sin\varphi' \cos(\omega' + \varepsilon), \\ \sqrt{m'}\,\alpha' &= \sin\varphi \sin(\omega + \varepsilon), & \sqrt{m'}\,\beta' &= \sin\varphi' \sin(\omega' + \varepsilon), \\ \sqrt{m''}\,\alpha'' &= \cos\varphi, & \sqrt{m''}\,\beta'' &= \cos\varphi'; \end{aligned}$$

it is plain that the last two of equations (2) will be satisfied, and the first three take the forms

$$\begin{aligned} \cos\gamma \cos\varphi + \sin\gamma \sin\varphi \cos\omega &= 0, \\ \cos\gamma \cos\varphi' + \sin\gamma \sin\varphi' \cos\omega' &= 0, \\ \cos\varphi \cos\varphi' + \sin\varphi \sin\varphi' \cos(\omega - \omega') &= 0. \end{aligned}$$



Hence, if the quadrantal spherical triangle ABC is constructed, and the arc $AD=\gamma$, having any arbitrary orientation on the sphere, drawn, and BD and CD joined, we shall have $\phi=BD$, $\phi'=CD$, $\omega=ADB$, $\omega'=ADC$ as the general solution of the system of equations (2).

Then, after substitution

$$T=\frac{dx^2+dy^2+dz^2}{2dt^2}+\frac{dx'^2+dy'^2+dz'^2}{2dt^2},$$

and if we put

$$\begin{aligned}
f&=\alpha'-\alpha'', & f'&=\alpha''-\alpha, & f''&=\alpha-\alpha',\\
g&=\beta'-\beta'', & g'&=\beta''-\beta, & g''&=\beta-\beta',\\
v&=x^2+y^2+z^2, & v'&=x'^2+y'^2+z'^2, & v''&=xx'+yy'+zz'
\end{aligned}$$

we have the expressions

$$\begin{aligned}
\Delta^2&=f^2v+2fgv''+g^2v',\\
\Delta'^2&=f'^2v+2f'g'v''+g'^2v',\\
\Delta''^2&=f''^2v+2f''g''v''+g''^2v'.
\end{aligned}$$

The equations of motion are now

$$\begin{aligned}
\frac{d^2x}{dt^2}&=\frac{\partial\Omega}{\partial x}, & \frac{d^2y}{dt^2}&=\frac{\partial\Omega}{\partial y}, & \frac{d^2z}{dt^2}&=\frac{\partial\Omega}{\partial z},\\
\frac{d^2x'}{dt^2}&=\frac{\partial\Omega}{\partial x'}, & \frac{d^2y'}{dt^2}&=\frac{\partial\Omega}{\partial y'}, & \frac{d^2z'}{dt^2}&=\frac{\partial\Omega}{\partial z'}.
\end{aligned}$$

Or, regarding Ω as a function of v, v', v'',

$$\begin{aligned}
\frac{d^2x}{dt^2}&=2x\frac{\partial\Omega}{\partial v}+x'\frac{\partial\Omega}{\partial v''}, & \frac{d^2x'}{dt^2}&=2x'\frac{\partial\Omega}{\partial v'}+x\frac{\partial\Omega}{\partial v''},\\
\frac{d^2y}{dt^2}&=2y\frac{\partial\Omega}{\partial v}+y'\frac{\partial\Omega}{\partial v''}, & \frac{d^2y'}{dt^2}&=2y'\frac{\partial\Omega}{\partial v'}+y\frac{\partial\Omega}{\partial v''},\\
\frac{d^2z}{dt^2}&=2z\frac{\partial\Omega}{\partial v}+z'\frac{\partial\Omega}{\partial v''}, & \frac{d^2z'}{dt^2}&=2z'\frac{\partial\Omega}{\partial v'}+z\frac{\partial\Omega}{\partial v''}.
\end{aligned}$$

From these by eliminating the partial derivatives of Ω, we obtain

$$\begin{aligned}
\frac{xd^2y-yd^2x}{dt^2}+\frac{x'd^2y'-y'd^2x'}{dt^2}&=0,\\
\frac{zd^2x-xd^2z}{dt^2}+\frac{z'd^2x'+x'd^2z'}{dt^2}&=0,\\
\frac{yd^2z-zd^2y}{dt^2}+\frac{y'd^2z'-z'd^2y'}{dt^2}&=0.
\end{aligned}$$

The integrals of which are

$$\frac{xdy - ydx}{dt} + \frac{x'dy' - y'dx'}{dt} = k \cos \mu,$$
$$\frac{zdx - xdz}{dt} + \frac{z'dx' - x'dz'}{dt} = k \sin \mu \cos \nu,$$
$$\frac{ydz - zdy}{dt} + \frac{y'dz' - z'dy'}{dt} = k \sin \mu \sin \nu,$$

k, μ and ν being the arbitrary constants. In addition there is the integral of living forces

$$T = \Omega + h.$$

If we put

$$u = \frac{dx^2 + dy^2 + dz^2}{dt^2}, \quad u' = \frac{dx'^2 + dy'^2 + dz'^2}{dt^2}, \quad u'' = \frac{dxdx' + dydy' + dzdz'}{dt^2}$$

it is evident that

$$\tfrac{1}{2}\frac{d^2v}{dt^2} - u = \frac{xd^2x + yd^2y + zd^2z}{dt^2},$$
$$\tfrac{1}{2}\frac{d^2v'}{dt^2} - u' = \frac{x'd^2x' + y'd^2y' + z'd^2z'}{dt^2},$$
$$\tfrac{1}{2}\frac{d^2v''}{dt^2} - u'' = \frac{xd^2x' + yd^2y' + zd^2z' + x'd^2x + y'd^2y + z'd^2z}{2dt^2}.$$

We put moreover

$$\rho = \frac{xdx' - x'dx + ydy' - y'dy + zdz' - z'dz,}{2dt}$$

whence

$$\frac{xdx' + ydy' + zdz'}{dt} = \tfrac{1}{2}\frac{dv''}{dt} + \rho,$$
$$\frac{x'dx + y'dy + z'dz}{dt} = \tfrac{1}{2}\frac{dv''}{dt} - \rho.$$

We have

$$\frac{d\rho}{dt} = \frac{xd^2x' - x'd^2x + yd^2y' - y'd^2y + zd^2z' - z'd^2z}{2dt^2}.$$

By the substitution of the values of $\frac{d^2x}{dt^2}$, ... , we obtain

$$\tfrac{1}{2}\frac{d^2v}{dt^2} - u = 2v \frac{\partial\Omega}{\partial v} + v''\frac{\partial\Omega}{\partial v''},$$
$$\tfrac{1}{2}\frac{d^2v'}{dt^2} - u' = 2v' \frac{\partial\Omega}{\partial v'} + v''\frac{\partial\Omega}{\partial v''},$$
$$\tfrac{1}{2}\frac{d^2v''}{dt^2} - u'' = v''\left(\frac{\partial\Omega}{\partial v} + \frac{\partial\Omega}{\partial v'}\right) + \tfrac{1}{2}(v + v')\frac{\partial\Omega}{\partial v''},$$
$$\frac{d\rho}{dt} = v''\left(\frac{\partial\Omega}{\partial v'} - \frac{\partial\Omega}{\partial v}\right) + \tfrac{1}{2}(v - v')\frac{\partial\Omega}{\partial v''}.$$

These equations take simpler forms when the variables are changed as follows:

$$\mathrm{w} = \tfrac{1}{4}(v + v'), \quad \mathrm{w}' = \tfrac{1}{4}(v - v'), \quad \mathrm{w}'' = \tfrac{1}{2}v'',$$
$$\nu = \tfrac{1}{2}(u + u'), \quad \nu' = \tfrac{1}{2}(u - u'), \quad \nu'' = u''.$$

Then they become

$$\left.\begin{aligned}
\frac{d^2\mathrm{w}}{dt^2} - \nu &= \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}} + \mathrm{w}'\,\frac{\partial\Omega}{\Delta\mathrm{w}'} + \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}''} = -\tfrac{1}{2}\,\Omega,\\
\frac{d^2\mathrm{w}'}{dt^2} - \nu' &= \mathrm{w}'\,\frac{\partial\Omega}{\partial\mathrm{w}} + \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}'},\\
\frac{d^2\mathrm{w}''}{dt^2} - \nu'' &= \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}} + \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}''},\\
\frac{d\rho}{dt} &= \mathrm{w}'\,\frac{\partial\Omega}{\partial\mathrm{w}''} - \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}'}.
\end{aligned}\right\} \qquad (3)$$

If we add to the first of these the equation of living force $\nu = \Omega + h$, we get

$$\frac{d^2\mathrm{w}}{dt^2} = \tfrac{1}{2}\,\Omega + h, \qquad (4)$$

an equation involving only the variables w, w′ and w″.

If we square the members of the three equations which constitute the principle of conservation of areas, and take their sum, the result will evidently be an equation which is not changed by a cyclical permutation of the letters x, y, z. We have the identical equation

$$\begin{aligned}(xdy - ydx)^2 + (zdx - xdz)^2 + (ydz - zdy)^2 \\ \equiv (x^2 + y^2 + z^2)(dx^2 + dy^2 + dz^2) - (xdx + ydy + zdz)^2,\end{aligned}$$

with the similar equation which is obtained by affixing accents to x, y, z. In addition there is the identity

$$\begin{aligned}(xdy - ydx)(x'dy' - y'dx') + (zdx - xdz)(z'dx' - x'dz') + (ydz - zdy)(y'dz' - z'dy') \\ \equiv (xx' + yy' + zz')(dxdx' + dydy' + dzdz') - (xdx' + ydy' + zdz')(x'dx + y'dy + z'dz).\end{aligned}$$

From these equations it will be seen that the equation of the sum of the squares takes the form

$$vu + v'u' + 2v''u'' - \frac{dv^2 + dv'^2 + 2dv''^2}{4dt^2} + 2\rho^2 = k^2,$$

or, after transforming into terms of the new variables, and, for convenience, writing k for $\tfrac{1}{2}k$,

$$\rho^2 = 2k^2 + \frac{d\mathrm{w}^2 + d\mathrm{w}'^2 + d\mathrm{w}''^2}{dt^2} - 2(\mathrm{w}\nu + \mathrm{w}'\nu' + \mathrm{w}''\nu''), \qquad (5)$$

an equation which is symmetrical.

It is evident now that, since the values of ν, ν', ν'' are known from the first three equations of (3), we shall have, as the equations determining w, w′ and w″, (4), (5) and the last of (3), provided we can find a relation connecting ρ with w, w′, w″, ν, ν', ν'' and the differentials of the first three.

Such a relation can be found in the following manner: Assume the four indeterminates X, X', X'', X''' so that the equations

$$xX + x'X' + \frac{dx}{dt}X'' + \frac{dx'}{dt}X''' = 0,$$
$$yX + y'X' + \frac{dy}{dt}X'' + \frac{dy'}{dt}X''' = 0,$$
$$zX + z'X' + \frac{dz}{dt}X'' + \frac{dz'}{dt}X''' = 0,$$

are satisfied; and treat the last as if they were equations of condition in the method of least squares, that is, multiply the first by x, the second by y, and the third by z, and take the sum for a first equation; and so on. In this way the normal equations formed from them are

$$vX + v''X' + \tfrac{1}{2}\frac{dv}{dt}X'' + \left(\tfrac{1}{2}\frac{dv''}{dt} + \rho\right)X''' = 0,$$
$$v''X + v'X' + \tfrac{1}{2}\left(\frac{dv''}{dt} - \rho\right)X'' + \tfrac{1}{2}\frac{dv'}{dt}X''' = 0,$$
$$\tfrac{1}{2}\frac{dv}{dt}X + \left(\tfrac{1}{2}\frac{dv''}{dt} - \rho\right)X' + uX'' + u''X''' = 0,$$
$$\left(\tfrac{1}{2}\frac{dv''}{dt} + \rho\right)X + \tfrac{1}{2}\frac{dv'}{dt}X' + u''X'' + u'X''' = 0.$$

As the number of these equations exceeds that of those from which they are derived, they are not independent, and the determinant, formed from the coefficients, vanishes; which is the condition determining ρ. This equation is

$$\begin{aligned}
&\left[\rho^2 + \frac{dvdv' - dv''^2}{4dt^2}\right]^2 + (vv' - v''^2)(uu' - u''^2) \\
&- \left[v'\rho^2 + \frac{v'dv'' - v''dv'}{dt}\rho + \frac{v'dv''^2 - 2v''dv'dv'' + vdv'^2}{4dt^2}\right]u \\
&- \left[v\rho^2 + \frac{v''dv - vdv''}{dt}\rho + \frac{vdv''^2 - 2v''dvdv'' + v'dv^2}{4dt^2}\right]u' \\
&+2\left[v''\rho^2 + \frac{v'dv - vdv'}{dt}\rho + \frac{(v'dv + vdv')dv'' - v''(dvdv' + dv''^2)}{4dt^2}\right]u'' = 0,
\end{aligned}$$

or, expressed in terms of the new variables,

$$\begin{aligned}
&\left[\rho^2 + \frac{d\text{w}^2 - d\text{w}'^2 - d\text{w}''^2}{dt^2}\right]^2 + 4(\text{w}^2 - \text{w}'^2 - \text{w}''^2)(\nu^2 - \nu'^2 - \nu''^2) \\
&-4[\text{w}\nu - \text{w}'\nu' - \text{w}''\nu'']\rho^2 \\
&+8\left[\frac{\text{w}'d\text{w}'' - \text{w}''d\text{w}'}{dt}\nu + \frac{\text{w}''d\text{w} - \text{w}d\text{w}''}{dt}\nu' + \frac{\text{w}d\text{w}' - \text{w}'d\text{w}}{dt}\nu''\right]\rho \\
&-4\left[\text{w}\,\frac{d\text{w}^2 + d\text{w}'^2 + d\text{w}''^2}{dt^2} - \frac{d\text{w}}{dt}\frac{d(\text{w}'^2 + \text{w}''^2)}{dt}\right]\nu \\
&-4\left[\text{w}'\,\frac{d\text{w}^2 + d\text{w}'^2 - d\text{w}''^2}{dt^2} - \frac{d\text{w}'}{dt}\frac{d(\text{w}^2 - \text{w}''^2)}{dt}\right]\nu' \\
&-4\left[\text{w}''\,\frac{d\text{w}^2 - d\text{w}'^2 + d\text{w}''^2}{dt^2} - \frac{d\text{w}''}{dt}\frac{d(\text{w}^2 - \text{w}'^2)}{dt}\right]\nu'' = 0.
\end{aligned}$$

If ρ^2 is eliminated from this equation by means of its value from (5), we shall have an equation of the first degree in ρ, from which the value of this quantity can be derived.

In *resumé*, we can present our results as follows: Let the five symbols Ω, ν, ν', ν'' and ρ have the significations

$$\Omega = [\mathrm{aw} + \mathrm{a'w'} + \mathrm{a''w''}]^{-\frac{1}{2}} + [\mathrm{bw} + \mathrm{b'w'} + \mathrm{b''w''}]^{-\frac{1}{2}} + [\mathrm{cw} + \mathrm{c'w'} + \mathrm{c''w''}]^{-\frac{1}{2}},$$

where a, b, c, ... , denote certain functions of the masses and of a single constant arbitrary quantity,

$$\begin{aligned}
\nu &= \frac{d^2\mathrm{w}}{dt^2} - \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}} - \mathrm{w}'\,\frac{\partial\Omega}{\partial\mathrm{w}'} - \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}''} = \frac{d^2\mathrm{w}}{dt^2} + \tfrac{1}{2}\,\Omega,\\
\nu' &= \frac{d^2\mathrm{w}'}{dt^2} - \mathrm{w}'\,\frac{\partial\Omega}{\partial\mathrm{w}} - \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}'},\\
\nu'' &= \frac{d^2\mathrm{w}''}{dt^2} - \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}} - \mathrm{w}\,\frac{\partial\Omega}{\partial\mathrm{w}''},
\end{aligned}$$

$$\rho = \frac{\left\{\begin{aligned}
&\left[\frac{d\mathrm{w}^2}{dt^2} - 2\mathrm{w}\nu + k^2\right]^2\\
&- [(\mathrm{w}\nu' - \mathrm{w}'\nu)^2 + (\mathrm{w}''\nu - \mathrm{w}\nu'')^2 - (\mathrm{w}''\nu' - \mathrm{w}'\nu'')^2]\\
&-2\left[\frac{d\mathrm{w}'}{dt}\,\frac{\mathrm{w}d\mathrm{w}' - \mathrm{w}'d\mathrm{w}}{dt} + \frac{d\mathrm{w}''}{dt}\,\frac{\mathrm{w}d\mathrm{w}'' - \mathrm{w}''d\mathrm{w}}{dt}\right]\nu\\
&-2\left[\frac{d\mathrm{w}}{dt}\,\frac{\mathrm{w}'d\mathrm{w} - \mathrm{w}d\mathrm{w}'}{dt} + \frac{d\mathrm{w}''}{dt}\,\frac{\mathrm{w}''d\mathrm{w}' - \mathrm{w}'d\mathrm{w}''}{dt}\right]\nu'\\
&-2\left[\frac{d\mathrm{w}}{dt}\,\frac{\mathrm{w}''d\mathrm{w} - \mathrm{w}d\mathrm{w}''}{dt} + \frac{d\mathrm{w}'}{dt}\,\frac{\mathrm{w}'d\mathrm{w}'' - \mathrm{w}''d\mathrm{w}'}{dt}\right]\nu''
\end{aligned}\right\}}{2\left[\frac{\mathrm{w}''d\mathrm{w}' - \mathrm{w}'d\mathrm{w}''}{dt}\,\nu + \frac{\mathrm{w}d\mathrm{w}'' - \mathrm{w}''d\mathrm{w}}{dt}\,\nu' + \frac{\mathrm{w}'d\mathrm{w} - \mathrm{w}d\mathrm{w}'}{dt}\,\nu''\right]}.$$

Then the differential equations, which determines w, w′ and w″, are

$$\begin{aligned}
\frac{d^2\mathrm{w}}{dt^2} &= \tfrac{1}{2}\,\Omega + h,\\
\rho^2 &= 2k^2 + \frac{d\mathrm{w}^2 + d\mathrm{w}'^2 + d\mathrm{w}''}{dt^2} - 2\,(\mathrm{w}\nu + \mathrm{w}'\nu' + \mathrm{w}''\nu''),\\
\frac{d\rho}{dt} &= \mathrm{w}'\,\frac{\partial\Omega}{\partial\mathrm{w}''} - \mathrm{w}''\,\frac{\partial\Omega}{\partial\mathrm{w}'}.
\end{aligned}$$

The first and second are of the second order, while the third is of the third order. It will be noticed that, although the expressions involved in them are not exactly symmetrical, yet they exhibit some approach to symmetry; and, perhaps, by a slight change in the notation, they may be made so. But I have not succeeded in discovering such a transformation.

MEMOIR No. 29.

On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon.

(Separately published, Cambridge, Mass., John Wilson & Son, pp. 28, 1877, Reprinted in *Acta Mathematica*, Vol. VIII, pp. 1-36, 1886.)

For more than sixty years after the publication of the *Principia*, astronomers were puzzled to account for the motion of the lunar perigee, simply because they could not conceive that terms of the second and higher orders, with respect to the disturbing force, produced more than half of it. For a similar reason, the great inequalities of Jupiter and Saturn remained a long time unexplained.

The rate of motion of the lunar perigee is capable of being determined from observation with about a thirteenth of the precision of the rate of mean motion in longitude. Hence if we suppose that the mean motion of the moon, in the century and a quarter which has elapsed since Bradley began to observe, is known within $3''$, it follows that the motion of the perigee can be got to within about 500,000th of the whole. None of the values hitherto computed from theory agrees as closely as this with the value derived from observation. The question then arises whether the discrepancy should be attributed to the fault of not having carried the approximation far enough, or is indicative of forces acting on the moon which have not yet been considered.

This question cannot be decisively answered until some method of computing the quantity considered is employed, which enables us to say, with tolerable security, that the neglected terms do not exceed a certain limit. If other forces besides gravity have a part in determining the positions of the heavenly bodies, the moon is unquestionably that one which will earliest exhibit traces of these actions; and the motion of the perigee is one of the things most likely to give us advice of them. Hence I propose, in this memoir, to compute the value of this quantity, so far as it depends on the mean motions of the sun and moon, with a degree of accuracy that shall leave nothing further to be desired.

Denoting the potential function by Ω, the differential equations of the moon, in rectangular coordinates, are

$$\frac{d^2x}{dt^2}=\frac{d\Omega}{dx}, \quad \frac{d^2y}{dt^2}=\frac{d\Omega}{dy}. \tag{1}$$

When terms, involving the solar eccentricity, are neglected, as is done here, it is known that these equations admit an integral,* the Eulerian multipliers for which are, respectively,

$$F=\frac{dx}{dt}+n'y, \quad G=\frac{dy}{dt}-n'x,$$

n' being the mean angular motion of the sun. When the equations are multiplied by these factors and the products added, it is seen that, not only is the resulting first member an exact derivative with respect to t, but that the second is also the exact derivative of Ω. Hence the integral is

$$\frac{dx^2+dy^2}{2dt^2}-n'\frac{xdy-ydx}{dt}=\Omega+C. \tag{2}$$

Let us now suppose that the lunar inequalities independent of the eccentricity, that is, those having the argument of the variation, have already been obtained, and that it is desired to get those which are multiplied by the simple power of this quantity: Denoting the latter by δx and δy, and, for convenience, putting

$$\frac{d^2\Omega}{dx^2}=H, \quad \frac{d^2\Omega}{dxdy}=J, \quad \frac{d^2\Omega}{dy^2}=K,$$

which will be all known functions of t, we shall have the linear differential equations

$$\frac{d^2\delta x}{dt^2}=H\delta x+J\delta y, \quad \frac{d^2\delta y}{dt^2}=K\delta y+J\delta x. \tag{3}$$

The Jacobian integral also, being subjected to the operation δ, furnishes another equation. Here we notice that when the arbitrary constant C is developed in ascending powers of e, only even powers present themselves, hence we have $\delta C=0$. In the equation, moreover, the partial derivatives of Ω may be replaced by their equivalents, the second differential quotients of the coördinates. Then, it is evident, the resulting equation may be written

$$F\frac{d\delta x}{dt}+G\frac{d\delta y}{dt}-\frac{dF}{dt}\delta x-\frac{dG}{dt}\delta y=0. \tag{4}$$

* As Jacobi was the first to announce this integral (*Comptes Rendus de l'Académie des Sciences de Paris*, Tom. III., p. 59), we shall take the liberty of calling it the Jacobian integral.

This is plainly an integral of equations (3) with the special value 0 attributed to the arbitrary constant. For taking the derivative of it with respect to t,

$$F\frac{d^2\delta x}{dt^2} + G\frac{d^2\delta y}{dt^2} - \frac{d^2F}{dt^2}\delta x - \frac{d^2G}{dt^2}\delta y = 0. \tag{5}$$

Hence the Eulerian multipliers, for obtaining (4) from (3), are, for the first equation, F, and for the second G. Making the multiplication and comparing the result with (5), we get the conditions

$$\frac{d^2F}{dt^2} = HF + JG, \quad \frac{d^2G}{dt^2} = KG + JF. \tag{6}$$

On comparing these with (3), we gather at once that the system of equations

$$\delta x = F, \quad \delta y = G,$$

is a particular solution of equations (3) and it also satisfies (4). This solution, being composed of terms having the same argument as the variation, is foreign to the solution we seek, and, in consequence, the arbitrary constant, multiplying it in the complete integrals of (3), must, for our problem, be supposed to vanish. But advantage may be taken of it to depress the order of the final equations obtained by elimination. For this purpose we adopt new variables ρ and σ, such that

$$\delta x = F\rho, \quad \delta y = G\sigma.$$

Relations (6) being considered, (3) and (4) then become

$$\begin{aligned}
&F\frac{d^2\rho}{dt^2} + 2\frac{dF}{dt}\frac{d\rho}{dt} + JG(\rho - \sigma) = 0,^*\\
&G\frac{d^2\sigma}{dt^2} + 2\frac{dG}{dt}\frac{d\sigma}{dt} + JF(\sigma - \rho) = 0,\\
&F^2\frac{d\rho}{dt} + G^2\frac{d\sigma}{dt} \qquad\qquad\qquad = 0.
\end{aligned}$$

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* Write this

$$\frac{d}{dt}\left(F^2\frac{d\rho}{dt}\right) + JFG(\rho - \sigma) = 0,$$

and put

$$\frac{d\rho}{dt} = F^{-2}\lambda, \quad \frac{d\sigma}{dt} = -G^{-2}\lambda.$$

If these values are substituted in the equation, after dividing by JFG and differentiating it, we get

$$\frac{d}{dt}\left[\frac{1}{JFG}\frac{d\lambda}{dt}\right] + \left[\frac{1}{F^2} + \frac{1}{G^2}\right]\lambda = 0.$$

In order to obtain an equation, from which the first derivative of the unknown shall be absent, put

$$\lambda = \sqrt{JFG}\,\mathrm{w}.$$

Then

$$\frac{d^2\mathrm{w}}{dt^2} + \Theta\mathrm{w} = 0.$$

(Note of 1886.)

If the value of σ is derived from the first and substituted in the third of these equations, the result will be

$$\frac{d^3\rho}{dt^3}+\frac{d}{dt}\left[\log\frac{F^3}{JG}\right]\frac{d^2\rho}{dt^2}+\left[\frac{J(F^2+G^2)}{FG}+\frac{JG}{F}\frac{d}{dt}\left(\frac{2}{JG}\frac{dF}{dt}\right)\right]\frac{d\rho}{dt}=0. \tag{7}$$

Let us now assume a variable w, such that

$$\frac{d\rho}{dt}=\sqrt{\frac{JG}{F}}\,\mathrm{w}.$$

The second term of (7) is removed by this transformation, and the equation takes the form of the reduced linear equation of the second order,

$$\frac{d^2\mathrm{w}}{dt^2}+\theta\mathrm{w}=0, \tag{8}$$

in which, after some reductions,

$$\theta=\frac{J(F^2+G^2)}{FG}+\frac{d^2.\log(JFG)}{2dt^2}-\left[\frac{d.\log(JFG)}{2dt}\right]^2. \tag{9}$$

It will be perceived that interchanging F and G produces no change in Θ: hence had we eliminated ρ instead of σ, the equation obtained would have been the same; and this is true in general,—we arrive always at the same value for Θ, no matter what variables may have been used to express the original differential equations. From this we may conclude that Θ depends only on the relative position of the moon with reference to the sun, and that it can be developed in a periodic series of the form

$$\theta_0+\theta_1\cos 2\tau+\theta_2\cos 4\tau+\ldots,$$

in which τ denotes the mean angular distance of the two bodies.

It may be noted also that Θ, as expressed above, does not involve the quantities H and K. It is obvious that, by means of the original differential equations, all second and higher derivatives may be eliminated from this expression, and that the Jacobian integral suffices for eliminating the first derivative of one of the variables. But it is not possible to express Θ as a function of the coördinates only without their derivatives.

II.

As the reduction of Θ, in the form just given, presents some difficulties, we will derive another from differential equations in terms of coördinates expressing the relative position of the moon to the sun.

Let the axes of rectangular coördinates have a constant velocity of rotation, so that the axis of x constantly passes through the centre of the sun, and adopt the imaginary variables

$$u = x + y\sqrt{-1}, \qquad s = x - y\sqrt{-1},$$

and put $\varepsilon^{\tau\sqrt{-1}} = \zeta$. In addition, let D denote the operation $-\frac{d}{d\tau}\sqrt{-1}$, so that

$$D(a\zeta^\nu) = \nu a\zeta^\nu,$$

and m denote the ratio of the synodic month to the sidereal year, or

$$\mathrm{m} = \frac{n'}{n - n'},$$

and μ being the sum of the masses of the earth and moon,

$$\varkappa = \frac{\mu}{(n - n')^2}.$$

Lastly, putting

$$\Omega = \frac{\varkappa}{\sqrt{us}} + \tfrac{3}{8}\mathrm{m}^2(u + s)^2, \tag{10}$$

the differential equations of motion are

$$\left.\begin{aligned} D^2u + 2\mathrm{m}Du + 2\frac{d\Omega}{ds} &= 0, \\ D^2s - 2\mathrm{m}Ds + 2\frac{d\Omega}{du} &= 0. \end{aligned}\right\} \tag{11}$$

Multiplying the first of these by Ds, the second by Du, adding the products and integrating the resulting equation, we have the Jacobian integral

$$DuDs + 2\Omega = 2C.$$

When the last three equations are subjected to the operation δ, the results are

$$\left.\begin{aligned} D^2\delta u + 2\mathrm{m}D\delta u + 2\frac{d^2\Omega}{duds}\delta u + 2\frac{d^2\Omega}{ds^2}\delta s &= 0, \\ D^2\delta s - 2\mathrm{m}D\delta s + 2\frac{d^2\Omega}{duds}\delta s + 2\frac{d^2\Omega}{du^2}\delta u &= 0, \\ DuD\delta s + DsD\delta u + 2\frac{d\Omega}{du}\delta u + 2\frac{d\Omega}{ds}\delta s &= 0. \end{aligned}\right\} \tag{12}$$

If, in these equations, the symbol δ is changed into D, they evidently still hold, since they then become the derivatives of the preceding equa-

tions. Hence the system of equations

$$\delta u = Du\,, \qquad \delta s = Ds\,.$$

forms a particular solution of them. For a like purpose as before, let us adopt new variables v and w, such that

$$\delta u = Du\,.\,v \qquad \delta s = Ds\,.\,w\,.$$

In terms of these, equations (12) become

$$\begin{aligned}
&Du\,.\,D^2v + 2\,[D^2u + \mathrm{m}Du]\,Dv + \left[D^3u + 2\mathrm{m}D^2u + 2\frac{d^2\Omega}{duds}\,Du\right]v + 2\frac{d^2\Omega}{ds^2}\,Ds\,.\,w = 0\,,\\
&Ds\,.\,D^2w + 2\,[D^2s - \mathrm{m}Ds]\,Dw + \left[D^3s - 2\mathrm{m}D^2s + 2\,\frac{d^2\Omega}{duds}\,Ds\right]w + 2\frac{d^2\Omega}{du^2}\,Du\,.\,v = 0\,,\\
&DuDs\,.\,D\,(v+w) + \left[DsD^2u + 2\,\frac{d\Omega}{du}\,Du\right]v + \left[DuD^2s + 2\frac{d\Omega}{ds}\,Ds\right]w = 0\,.
\end{aligned}$$

If the second and third derivatives of u and s are eliminated from these equations by means of equations (11), we get

$$\left.\begin{aligned}
&Du\,.\,D^2v - 2\left[2\frac{d\Omega}{ds} + \mathrm{m}Du\right]Dv - 2\frac{d^2\Omega}{ds^2}\,Ds\,.\,(v-w) = 0\,,\\
&Ds\,.\,D^2w - 2\left[2\frac{d\Omega}{du} - \mathrm{m}Ds\right]Dw - 2\frac{d^2\Omega}{du^2}\,Du\,.\,(w-v) = 0\,,\\
&DuDs\,.\,D\,(v+w) - 2\left[\frac{d\Omega}{ds}\,Ds - \frac{d\Omega}{du}\,Du + \mathrm{m}DuDs\right](v-w) = 0\,.
\end{aligned}\right\} \qquad (13)$$

If the first of these equations is multiplied by Ds the second by Du, and the products added, the resulting equation will evidently be the derivative of the third; but if the products are subtracted, the second from the first, we get

$$\begin{aligned}
DuDs\,.\,D^2\,(v-w) - 2D\Omega\,.\,D\,(v-w) - 2\left[\frac{d\Omega}{ds}\,Ds - \frac{d\Omega}{du}\,Du + \mathrm{m}DuDs\right]D\,(v+w)&\\
-2\left[\frac{d^2\Omega}{du^2}\,Du^2 + \frac{d^2\Omega}{ds^2}\,Ds^2\right](v-w) = 0\,.&
\end{aligned}$$

For brevity we will write

$$\Delta = \frac{d\Omega}{ds}\,Ds - \frac{d\Omega}{du}\,Du + \mathrm{m}DuDs\,,$$

and put

$$\rho = v + w, \qquad \sigma = v - w,$$

then the last two equations, which will be those employed for the solution of the problem, become

$$\left.\begin{aligned}
&DuDs\,.\,D\rho - 2\Delta\,.\,\sigma = 0,\\
&D\,[DuDs\,.\,D\sigma] - 2\Delta\,.\,D\rho - 2\left[\frac{d^2\Omega}{du^2}\,Du^2 + \frac{d^2\Omega}{ds^2}\,Ds^2\right]\sigma = 0.
\end{aligned}\right\} \qquad (14)$$

Eliminating $D\rho$ between these equations, a single equation involving only the unknown σ, is obtained,

$$D\,[DuDs\,.\,D\sigma] - 2\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2 + \frac{2\Delta^2}{DuDs}\right]\sigma = 0. \tag{15}$$

In order to remove the term involving $D\sigma$, a last transformation will be made; we put

$$\sigma = \frac{\mathrm{w}}{\sqrt{DuDs}}.$$

Then the differential equation, determining w, is

$$D^2\mathrm{w} = \theta\mathrm{w},$$

in which

$$\theta = \frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right] + \left(\frac{2\Delta}{DuDs}\right)^2 + \frac{D^2(DuDs)}{2DuDs} - \left[\frac{D\,(DuDs)}{2DuDs}\right]^2$$

$$= \frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right] + \left(\frac{2\Delta}{DuDs}\right)^2 - \frac{D^2\Omega}{DuDs} - \left[\frac{D\Omega}{DuDs}\right]^2.$$

But we have

$$D\Omega = \frac{d\Omega}{du}Du + \frac{d\Omega}{ds}Ds,$$

$$D^2\Omega = \frac{d^2\Omega}{du^2}Du^2 + 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2 + 2\mathrm{m}\Delta - 2\mathrm{m}^2DuDs - 4\frac{d\Omega}{du}\frac{d\Omega}{ds},$$

in which, from the latter equation, have been eliminated the second derivatives of u and s, by means of their values obtained from equations (11). From these is obtained

$$D^2\Omega + \frac{[D\Omega]^2}{DuDs} = \frac{d^2\Omega}{du^2}Du^2 + 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2 + \frac{\Delta^2}{DuDs} - \mathrm{m}^2DuDs,$$

on substitution of which in the value of Θ, there results

$$\theta = \frac{1}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 - 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2\right] + 3\left(\frac{\Delta}{DuDs}\right)^2 + \mathrm{m}^2. \tag{16}$$

The partial derivatives of Ω, involved in this expression, have the values

$$\frac{d\Omega}{du} = -\tfrac{1}{2}\frac{\kappa}{r^3}s + \tfrac{3}{4}\mathrm{m}^2(u+s),$$

$$\frac{d\Omega}{ds} = -\tfrac{1}{2}\frac{\kappa}{r^3}u + \tfrac{3}{4}\mathrm{m}^2(u+s),$$

$$\frac{d^2\Omega}{du^2} = \tfrac{3}{4}\frac{\kappa}{r^5}s^2 + \tfrac{3}{4}\mathrm{m}^2,$$

$$\frac{d^2\Omega}{duds} = \tfrac{1}{4}\frac{\kappa}{r^3} + \tfrac{3}{4}\mathrm{m}^2,$$

$$\frac{d^2\Omega}{ds^2} = \tfrac{3}{4}\frac{\kappa}{r^5}u^2 + \tfrac{3}{4}\mathrm{m}^2,$$

where, for us, has been written r^2, the square of the moon's radius vector. After the substitution of these, it will be found that we can write

$$\Theta = \frac{x}{r^3} + \frac{3}{8}\,\frac{\frac{x}{r^5}[uDs - sDu]^2 + \mathrm{m}^2(Du - Ds)^2}{C - \Omega} + \frac{3}{4}\left[\frac{\Delta}{C-\Omega}\right]^2 + \mathrm{m}^2, \tag{17}$$

in which

$$\Delta = \left[-\tfrac{1}{2}\frac{x}{r^3} + \tfrac{9}{4}\mathrm{m}^2\right][uDs - sDu] - \tfrac{3}{4}\mathrm{m}^2(uDu - sDs) + 2\mathrm{m}(C - \Omega).$$

This expression for Θ, from which all derivatives of u and s, higher than the first, have been eliminated whenever they presented themselves, is suitable for development in infinite series, when the method of special values is employed. The quadrant being divided into a certain number of equal parts with reference to τ, we compute the values of the four variables u, s, Du, Ds, of which Θ is a function, for these special values of τ, and by substitution ascertain the corresponding values of Θ. From the last, by the well-known process, are derived the several coefficients of the periodic terms of Θ. A discussion of the lunar inequalities, which are independent of everything but the parameter m, shows that the values of u and s have the form

$$u = \Sigma_i \,.\, \mathrm{a}_i\zeta^{2i+1}, \qquad s = \Sigma_i \,.\, \mathrm{a}_i\zeta^{-2i-1},$$

where i receives all integral values from $-\infty$ to $+\infty$, zero included, and the coefficients a_i are constant, being equivalent each to the same constant multiplied by a function of m which is of the $2i$th order with respect to this parameter.

By taking the derivatives

$$Du = \Sigma_i \,.\, (2i+1)\,\mathrm{a}_i\zeta^{2i+1}, \qquad Ds = -\Sigma_i \,.\, (2i+1)\,\mathrm{a}_i\zeta^{-2i-1}.$$

It will be seen from these equations that, in the terms were i is large, we will be subjected to the inconvenience of having the errors, with which the coefficients a_i are necessarily affected, multiplied by large numbers. This will be avoided by employing, in the computation of Θ, the formula

$$uDs - sDu = 2\mathrm{m}r^2 - \tfrac{3}{2}\mathrm{m}^2D^{-1}(u^2 - s^2),$$

where D^{-1} denotes the inverse operation of D. This does not give the constant term of $uDs - sDu$, but this can be obtained from the expression

$$-2\Sigma_i \,.\, (2i+1)\,\mathrm{a}_i^2,$$

which is not subject to the difficulty mentioned above. Wherever Du and Ds occur elsewhere in the formula for Θ, they are multiplied by the small factor m^2, and, in consequence, the given formulas suffice.

This mode of proceeding will give only a numerical result: if we wish to have m left indeterminate in the development of Θ, it will be advantageous to give the latter another form. In this case there is no objection to the appearance of second and third derivatives of u and s in the expression of Θ.

From the value of $D^2\Omega$, previously given, it is easy to conclude that

$$\frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2+\frac{d^2\Omega}{ds^2}Ds^2\right]=-4\frac{d^2\Omega}{duds}-2\left(\frac{\Delta}{DuDs}\right)^2+2\mathrm{m}^2-\frac{D^2(DuDs)}{DuDs}+\tfrac{1}{2}\left[\frac{D(DuDs)}{DuDs}\right]^2.$$

If this is substituted in the expression first given for Θ, and we note that

$$4\frac{d^2\Omega}{duds}=\frac{\varkappa}{r^3}+3\mathrm{m}^2,$$

$$\Delta=\tfrac{1}{2}[DuD^2s-DsD^2u]-\mathrm{m}DuDs,$$

the latter being obtained by substituting in the previously given value of Δ, the values of the partial derivatives of Ω given by equations (11), we get

$$\Theta=-\left[\frac{\varkappa}{r^3}+\mathrm{m}^2\right]+2\left[\tfrac{1}{2}\left(\frac{D^2u}{Du}-\frac{D^2s}{Ds}\right)+\mathrm{m}\right]^2$$
$$-\left[\tfrac{1}{2}\left(\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right)\right]^2-D\left[\tfrac{1}{2}\left(\frac{D^2u}{Du}+\frac{D^2s}{Ds}\right)\right]. \qquad (18)$$

For the development of the first term of this expression, we can employ either of the following equations which result from equations (11),

$$\frac{\varkappa}{r^3}+\mathrm{m}^2=\frac{D^2u+2\mathrm{m}Du+\frac{3}{2}\mathrm{m}^2s}{u}+\tfrac{5}{2}\mathrm{m}^2$$
$$=\frac{D^2s-2\mathrm{m}Ds+\frac{3}{2}\mathrm{m}^2u}{s}+\tfrac{5}{2}\mathrm{m}^2,$$

which, if one studies symmetry of expression, may be written

$$\frac{\varkappa}{r^3}+\mathrm{m}^2=\left[\frac{Du}{u}+\mathrm{m}\right]^2+D\left[\frac{Du}{u}+\mathrm{m}\right]+\tfrac{3}{2}\mathrm{m}^2\left[1+\frac{s}{u}\right]$$
$$=\left[\frac{Ds}{s}-\mathrm{m}\right]^2+D\left[\frac{Ds}{s}-\mathrm{m}\right]+\tfrac{3}{2}\mathrm{m}^2\left[1+\frac{u}{s}\right]$$

and if half the sum of the second members is substituted for the first term in (18) we shall have a singularly symmetrical expression for Θ.

If the values of u and s in terms of ζ are substituted in the first of these equations, we get

$$\frac{\varkappa}{r^3}+\mathrm{m}=1+2\mathrm{m}+\tfrac{5}{2}\mathrm{m}^2+\frac{\Sigma_i\,.\,[4i(i+1+\mathrm{m})\,\mathrm{a}_i+\frac{3}{2}\mathrm{m}^2\mathrm{a}_{-i-1}]\,\zeta^{2i}}{\Sigma_i\,.\,\mathrm{a}_i\zeta^{2i}}.$$

Let the last term of the second member of this equation be denoted by the series

$$\Sigma_i \,.\, R_i \zeta^{2i};$$

since r is a series of cosines, we must have, in consequence of the equations of condition which the a_i satisfy $R_{-i} = R_i$, and the equations, which determine these coefficients, can be obtained from the formula.

$$\Sigma_j \,.\, a_{i-j} R_j = 4i\,(i+1+m)\, a_i + \tfrac{9}{2} m^2 a_{-i-1},$$

when we attribute to i, in succession, all integral values from $i=0$ to $i=\infty$, or which is preferable, from $i=0$ to $i=-\infty$. The following are all the equations and terms which need be retained when it is proposed to neglect quantities of the same order of smallness as m^{10};

$$\begin{aligned}
a_0 R_0 + (a_1 + a_{-1}) R_1 + (a_2 + a_{-2}) R_2 &= \tfrac{9}{2} m^2 a_{-1},\\
a_{-1} R_0 + (a_0 + a_{-2}) R_1 + a_1 R_2 &= -4m a_{-1} + \tfrac{9}{2} m^2 a_0,\\
a_{-2} R_0 + (a_{-1} + a_{-3}) R_1 + a_0 R_2 + a_1 R_3 &= 8(1-m)\, a_{-2} + \tfrac{9}{2} m^2 a_1,\\
a_{-2} R_1 + a_{-1} R_2 + a_0 R_3 &= 12(2-m)\, a_{-3} + \tfrac{9}{2} m^2 a_2,\\
a_{-3} R_1 + a_{-2} R_2 + a_{-1} R_3 + a_0 R_4 &= 16(3-m)\, a_{-4} + \tfrac{9}{2} m^2 a_3.
\end{aligned}$$

For the purpose of illustrating the present method, we content ourselves with giving the following approximate formula:—

$$\begin{aligned}
\frac{\kappa}{r^3} + m^2 = 1 + 2m + \tfrac{3}{2} m &- \tfrac{9}{2} m^2 a_1 + 4m a_{-1}(a_1 + a_{-1})\\
&+ [\tfrac{3}{2} m^2 - 4m a_{-1}](\zeta^2 + \zeta^{-2}) + [8(1-m)\, a_{-2} + \tfrac{9}{2} m^2 (a_1 - a_{-1}) + 4m a_{-1}^2](\zeta^4 + \zeta^{-4}),
\end{aligned}$$

where, for convenience in writing, it has been assumed that $a_0 = 1$, and consequently that a_i denotes here the ratio to a_0, which, as has been mentioned above, is a function of m. The absolute term and the coefficient of $\zeta^4 + \zeta^{-4}$ are affected with errors of the eighth order, while the coefficient of $\zeta^2 + \zeta^{-2}$ is affected with one of the sixth order.

We attend now to the remaining terms of Θ. If we put

$$\frac{D^2 u}{Du} = \frac{\Sigma_i \,.\, (2i+1)^2 a_i \zeta^{2i}}{\Sigma_i \,.\, (2i+1)\, a_i \zeta^{2i}} = \Sigma_i \,.\, U_i \zeta^{2i},$$

it is plain that we shall have

$$\frac{D^2 s}{Ds} = -\frac{\Sigma_i \,.\, (2i+1)^2 a_i \zeta^{-2i}}{\Sigma_i \,.\, (2i+1)\, a_i \zeta^{-2i}} = -\Sigma_i \,.\, U_i \zeta^{-2i},$$

and in consequence,

$$\begin{aligned}
\tfrac{1}{2}\left(\frac{D^2 u}{Du} - \frac{D^2 s}{Ds}\right) &= \Sigma_i \,.\, \tfrac{1}{2}(U_i + U_{-i})\, \zeta^{2i},\\
\tfrac{1}{2}\left(\frac{D^2 u}{Du} + \frac{D^2 s}{Ds}\right) &= \Sigma_i \,.\, \tfrac{1}{2}(U_i - U_{-i})\, \zeta^{2i}.
\end{aligned}$$

From this it will be seen that the development of $\frac{D^2 u}{Du}$ will suffice for obtaining all the remaining terms of Θ. Let us put

$$h = (2i+1)\,\mathrm{a}_i\,.$$

The equations which determine the coefficients U_i are given by the formula

$$\Sigma_j\,.\,h_{i-j}U_j = (2i+1)\,h_i\,,$$

but, in order to exhibit some of their properties, I write a few, *in extenso*, thus:

$$\left.\begin{array}{l}
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots + h_0 U_{-2} + h_{-1} U_{-1} + h_{-2}(U_0 - 1) + h_{-3} U_1 + h_{-4} U_2 + \ldots = -4h_{-2}, \\
\ldots + h_1 U_{-2} + h_0\ U_{-1} + h_{-1}(U_0 - 1) + h_{-2} U_1 + h_{-3} U_2 + \ldots = -2h_{-1}, \\
\ldots + h_2 U_{-2} + h_1\ U_{-1} + h_0\ (U_0 - 1) + h_{-1} U_1 + h_{-2} U_2 + \ldots = \quad 0, \\
\ldots + h_3 U_{-2} + h_2\ U_{-1} + h_1\ (U_0 - 1) + h_0\ U_1 + h_{-1} U_2 + \ldots = \quad 2h_1, \\
\ldots + h_4 U_{-2} + h_3\ U_{-1} + h_2\ (U_0 - 1) + h_1\ U_1 + h_0\ U_2 + \ldots = \quad 4h_2, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} \qquad (19)$$

When the subscripts of both the h and U in these equations are negatived, and the signs of the right-hand members reversed, the system of equations is the same as before. Hence, if we have found the value of U_i, which is a function of the h, the value of U_{-i} will be got from it by simply negativing the subscripts of all the h involved in it and reversing the sign of the whole expression. When this operation is applied to the particular unknown $U_0 - 1$, we get the condition

$$U_0 - 1 = -(U_0 - 1);$$

whence we have, rigorously,

$$U_0 = 1\,.$$

This result can also be established by the aid of a definite integral. The absolute term, in the development of $\frac{D^{\nu+1}u}{D^\nu u}$ in powers of ζ, is given by the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{D^{\nu+1}u}{D^\nu u}\,d\tau = \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi}\frac{\frac{d^{\nu+1}u}{d\tau^{\nu+1}}}{\frac{d^\nu u}{d\tau^\nu}}\,d\tau.$$

The indefinite integral of the expression under the sign of integration is

$$\log\frac{d^\nu u}{d\tau^\nu} = \log\left[\frac{d^\nu x}{d\tau^\nu} + \frac{d^\nu y}{d\tau^\nu}\sqrt{-1}\right],$$

and if, for the moment, we take ρ and ϕ such that

$$\frac{d^\nu x}{d\tau^\nu} = \rho \cos\varphi, \quad \frac{d^\nu y}{d\tau^\nu} = \rho \sin\varphi,$$

this integral takes the shape

$$\log\rho + \varphi\sqrt{-1}.$$

The first term of this has the same value for $\tau = 0$ and $\tau = 2\pi$, and consequently contributes nothing to the value of the definite integral. Thus we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{D^{\nu+1}u}{D^\nu u}\, d\tau = \frac{1}{2\pi}\,[\varphi]_{\tau=0}^{\tau=2\pi}.$$

When $\tau = 0$, let ϕ be assumed between 0 and 2π: it will be found that ϕ has the value 0 or $\frac{\pi}{2}$ or π or $\frac{3}{2}\pi$ according as ν is of the form 4μ or $4\mu + 1$ or $4\mu + 2$ or $4\mu + 3$. Moreover, when τ augments, ϕ also augments, and when τ has passed over one circumference, ϕ has also augmented by a circumference. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{D^{\nu+1}u}{D^\nu u}\, d\tau = 1.$$

It follows, therefore, that ν denoting zero or a positive integer, the absolute term of the development of $\frac{D^{\nu+1}u}{D^\nu u}$ in integral powers of ζ is 1. And, in like manner, the absolute term of $\frac{D^{\nu+1}s}{D^\nu s}$ is -1.

Equations (19) are readily solved by successive approximations, and when terms of the tenth order are neglected, we can write

$$\begin{aligned}
\frac{D^2u}{Du} = 1 &+ 2\,[\,h_1 - h_{-1}h_2 + h_1 h_1 h_{-1}]\,\zeta^2 \\
&- 2\,[\,h_{-1} - h_1 h_{-2} + h_{-1}h_{-1}h_1\,]\,\zeta^{-2} \\
&+ 2\,[2h_2 - h_1 h_1 - 2h_{-1}h_3 + 4h_1 h_{-1}h_2 - 2h_1 h_1 h_1 h_{-1}]\,\zeta^4 \\
&- 2\,[2h_{-2} - h_{-1}h_{-1} - 2h_1 h_{-3} + 4h_{-1}h_1 h_{-2} - 2h_{-1}h_{-1}h_{-1}h_1\,]\,\zeta^{-4} \\
&+ 2\,[3h_3 - 3h_1 h_2 + h_1 h_1 h_1\,]\,\zeta^6 \\
&- 2\,[3h_{-3} - 3h_{-1}h_{-2} + h_{-1}h_{-1}h_{-1}]\,\zeta^{-6} \\
&+ 2\,[4h_4 - 4h_1 h_3 + 4h_1 h_1 h_2 - 2h_2 h_2 - h_1 h_1 h_1 h_1\,]\,\zeta^8 \\
&- 2\,[4h_{-4} - 4h_{-1}h_{-3} + 4h_{-1}h_{-1}h_{-2} - 2h_{-2}h_{-2} - h_{-1}h_{-1}h_{-1}h_{-1}]\,\zeta^{-8},
\end{aligned}$$

where we have supposed again that $h_0 = a_0 = 1$.

With the same degree of approximation we have used for $\frac{\varkappa}{r^3} + m^2$, Θ can be written

$$\begin{aligned}\theta = 1 + 2m - \tfrac{1}{2} m^2 + \tfrac{3}{2} m^2 a_1 + 54 a_1^2 + (12 - 4m)\, a_1 a_{-1} + (6 - 4m)\, a_{-1}^2 \\ + [(6 + 12m)\, a_1 + (6 + 8m)\, a_{-1} - \tfrac{3}{2} m^2] (\zeta^2 + \zeta^{-2}) \\ + [20m a_2 + (16 + 20m)\, a_{-2} - (9 + 40m)\, a_1^2 + 6 a_1 a_{-1} + (7 + 4m)\, a_{-1}^2 \\ - \tfrac{3}{2} m^2 (a_1 - a_{-1})] (\zeta^4 + \zeta^{-4}).\end{aligned}$$

In the determination of the terms of the lunar coördinates which depend only on the parameter m, it has been found that, with errors of the sixth order,

$$a_1 = \tfrac{3}{16} \frac{6 + 12m + 9m^2}{6 - 4m + m^2} m^2,$$

$$a_{-1} = - \tfrac{3}{16} \frac{38 + 28m + 9m^2}{6 - 4m + m^2} m^2,$$

and, with errors of the eighth order,

$$a_2 = \tfrac{27}{256} \frac{2 + 4m + 3m^2}{[6 - 4m + m^2][30 - 4m + m^2]} \left[238 + 40m + 9m^2 - 32 \frac{29 - 35m}{6 - 4m + m^2} \right] m^4,$$

$$a_{-2} = \tfrac{27}{64} \frac{2 + 4m + 3m^2}{[6 - 4m + m^2][30 - 4m + m^2]} \left[- 28 - 7m + 24 \frac{7 - m}{6 - 4m + m^2} \right] m^4.*$$

No use will be made of these formulas in the sequel of this memoir: they are given only that we may at need easily deduce an approximate literal expansion for the important function Θ.

III.

In the preceding discussion it has been established that the determination of the lunar inequalities, which have the simple power of the eccentricity as factor, depends on the integration of the linear differential equation

$$D^2 w = \theta w;$$

to the treatment of which we accordingly proceed. We assume that the development of Θ, in a series of the form

$$\theta = \Sigma_i \,.\, \theta_i \zeta^{2i},$$

has been obtained. Here we have the condition $\Theta_{-i} = \Theta_i$. If Θ_1, Θ_2, &c., are, to a considerable degree, smaller than Θ_0, an approximate statement of the equation is

$$D^2 w = \theta_0 w;$$

* These expressions will be established in another memoir.

the complete integral of which is

$$\mathrm{w} = K\zeta^{\mathrm{c}} + K'\zeta^{-\mathrm{c}},$$

K and K' being the arbitrary constants and c being written for $\sqrt{\Theta_0}$. When the additional terms of Θ are considered, the effect is to modify this value of c, and also to add to w new terms of the general form $A\zeta^{\pm \mathrm{c}+2i}$. It is plain, therefore, that we may suppose

$$\mathrm{w} = K\mathrm{f}(\zeta, \mathrm{c}) + K'\mathrm{f}(\zeta, -\mathrm{c}),$$

and may take, as a particular integral,

$$\mathrm{w} = \Sigma_i \,.\, \mathrm{b}_i \zeta^{\mathrm{c}+2i},$$

b_i being a constant coefficient. If this equivalent of w is substituted in the differential equation, we get the equation

$$[\mathrm{c} + 2j]^2 \,\mathrm{b}_j - \Sigma_i \,.\, \theta_{j-i} \mathrm{b}_i = 0\,, \tag{20}$$

which holds for all integral values for j, positive and negative. These conditions determine the ratios of all the coefficients b_i to one of them, as b_0, which may then be regarded as the arbitrary constant. They also determine c, which is the ratio of the synodic to the anomalistic month. For the purpose of exhibiting more clearly the properties of the equations represented generally by (20), I write a few of them *in extenso:* for convenience let

$$[i] = (\mathrm{c} + 2i)^2 - \theta_0\,;$$

then

$$\left.\begin{array}{llllll}
\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot \\
\ldots + [-2]\,\mathrm{b}_{-2} & - \theta_1 \mathrm{b}_{-1} & - \theta_2 \mathrm{b}_0 & - \theta_3 \mathrm{b}_1 & - \theta_4 \mathrm{b}_2 & - \ldots = 0, \\
\ldots - \theta_1 \mathrm{b}_{-2} & + [-1]\,\mathrm{b}_{-1} & - \theta_1 \mathrm{b}_0 & - \theta_2 \mathrm{b}_1 & - \theta_3 \mathrm{b}_2 & - \ldots = 0, \\
\ldots - \theta_2 \mathrm{b}_{-2} & - \theta_1 \mathrm{b}_{-1} & + [0]\,\mathrm{b}_0 & - \theta_1 \mathrm{b}_1 & - \theta_2 \mathrm{b}_2 & - \ldots = 0, \\
\ldots - \theta_3 \mathrm{b}_{-2} & - \theta_2 \mathrm{b}_{-1} & - \theta_1 \mathrm{b}_0 & + [1]\,\mathrm{b}_1 & - \theta_1 \mathrm{b}_2 & - \ldots = 0, \\
\ldots - \theta_4 \mathrm{b}_{-2} & - \theta_3 \mathrm{b}_{-1} & - \theta_2 \mathrm{b}_0 & - \theta_1 \mathrm{b}_1 & + [2]\,\mathrm{b}_2 & - \ldots = 0, \\
\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot
\end{array}\right\} \tag{21}$$

If, from this group of equations, infinite in number, and the number of terms in each equation also infinite, we eliminate all the b except one, we get a symmetrical determinant involving c, which, equated to zero, determines this quantity. This equation we will denote thus:—

$$\mathfrak{D}(\mathrm{c}) = 0\,. \tag{22}$$

If, in (20), we put $-\mathrm{c}$ for c, $-j$ for j, and suppose that b_j is now denoted by b_{-j}, the equation is the same as at first; hence the determinant

just mentioned remains unchanged, when for c in it we substitute $-$ c, and

$$\mathfrak{D}(-c) = \mathfrak{D}(c),$$

or, in other words, $\mathfrak{D}(c)$ is a function of c^2. Again, in the same equation, let $c + 2\nu$ be substituted for c, ν being any positive or negative integer, and write $j - \nu$ for j, and suppose that b_j is now denoted by $b_{j+\nu}$. The equation is again the same as at first, and hence the determinant suffers no change when $c + 2\nu$ is written in it for c. That is,

$$\mathfrak{D}(c + 2\nu) = \mathfrak{D}(c).$$

It follows from all this that if (22) is satisfied by a root $c = c_0$, it will also have, as roots, all the quantities contained in the expression

$$\pm c_0 + 2i,$$

where i denotes any positive or negative integer or zero. And these are all the roots the equation admits; for each of the expressions denoted by $[i]$ is of two dimensions in c, and may be regarded as introducing into the equation the two roots $2i + c_0$ and $2i - c_0$. Consequently the roots are either all real or all imaginary, and it is impossible that the equation should have any equal roots unless all the roots are integral. But in the last case the inequalities we treat would evidently coalesce with those having the argument of the variation, and could not be separated from them; hence this case may be set aside as practically not occurring.

It is evident from the foregoing remarks that, in an analytical point of view, it is indifferent which of the roots of (22) is taken as the value of c; in every case we get the same value for w. For, denoting the mean anomaly of the moon by ξ, we have the infinite series of arguments

$$\ldots \xi - 4\tau, \quad \xi - 2\tau, \quad \xi, \quad \xi + 2\tau, \quad \xi + 4\tau \ldots$$

each of which can be made to play the same *rôle* as ξ, and analysis knows no distinction between them. Hence the equation, which determines the motion of ξ, must, of necessity, also give the motions of all the arguments of the series above, as well as of their negatives.* One has, however, been in the habit of taking for c the root which approximates to $\sqrt{\Theta_0}$.

* A similar condition of things occurs in many less complex problems; for instance, in the determination of the principal axes of rotation of a rigid body. Although there is but one set of such axes, yet the final equation, solving the question, is of the third degree, all because analysis knows no distinction between the axes of x, y, and z.

It may be well to notice here the modifications which the addition to the investigation of terms of higher orders produces in equation (22). This may be written

$$\Pi(x \pm c_0 + 2i) = 0,$$

where x is the unknown quantity and Π is a symbol denoting the product of the infinite number of factors obtained by attributing to i all integral values positive and negative, zero included, and taking in succession the ambiguous sign in both significations. Had the terms, involving higher powers of e, been included in the investigation, the equation would have been

$$\Pi(x + jc_0 + 2i) = 0,$$

where j receives all integral values positive and negative. If, furthermore, we had included all terms involving the argument τ and its odd multiples, the equation would have been

$$\Pi(x + jc_0 + i) = 0.$$

If to these we had added all terms depending on the solar eccentricity, the equation would have been

$$\Pi(x + jc_0 + i + km) = 0,$$

where k is also to receive all integral values positive and negative.

A similar thing is true in the general planetary problem. Professor Newcomb says,* "The quantities b," where b is of similar signification with c_0 above, "ought, perhaps, to appear as the roots of an equation of the $3n^{\text{th}}$ degree." But it is plain, from the foregoing remarks, that not only does this equation contain the $3n$ roots $b_1, b_2 \ldots . b_{3n}$, but also every root given by the general integral linear function of the b

$$i_1 b_1 + i_2 b_2 + \ldots + i_{3n} b_{3n},$$

for which, in the analysis, the corresponding argument

$$i_1 \lambda_1 + i_2 \lambda_2 + \ldots + i_{3n} \lambda_{3n}$$

can play the same rôle as any one of the individual arguments λ. Hence this equation, in all cases but the problem of two bodies, must be regarded as transcendental or of infinite degree.

The equations which determine the coefficients b_i and the quantity c, having the form of normal equations in the method of least squares, can be

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* On the General Integrals of Planetary Motion, Smithsonian Contributions to Knowledge, No. 281, p. 31.

solved by the process usually adopted for the latter. Let two of these equations be written

$$[j]\,b_j - \Sigma_i\,.\,\theta_{j-i} b_i = 0,$$
$$[\nu]\,b_\nu - \Sigma_i\,.\,\theta_{\nu-i} b_i = 0,$$

where, in the first, the summation does not include the value $i = j$, or in the second the value $i = \nu$. The result of the elimination of b_ν from these is

$$\left[[j] - \frac{\theta_{j-\nu}\theta_{j-\nu}}{[\nu]}\right] b_j - \Sigma_i\,.\left[\theta_{j-i} + \frac{\theta_{j-\nu}\theta_{i-\nu}}{[\nu]}\right] b_i = 0,$$

where, in the summation, i does not receive the values j and ν. This equation may be written

$$[j]^{(\nu)}\,b_j - \Sigma_i\,.\,\theta^{(\nu)}_{j-i}\,b_i = 0.$$

In like manner we may eliminate from the system of equations a second unknown $b_{\nu'}$. And the general form of equation obtained may be written

$$[j]^{(\nu,\nu')}\,b_j - \Sigma_i\,.\,\theta^{(\nu,\nu')}_{j-i}\,b_i = 0,$$

where, in the summation, i receives neither of the values j, ν and ν'. This process may be continued until all the b, having sensible values but b_0, are eliminated; and the single equation remaining, after division by b_0, may be written

$$[0]^{(\ldots -2, -1, 1, 2 \ldots)} = 0.$$

This determines c: when we pursue the method of numerical substitutions, it will be the most advantageous course to perform the preceding elimination twice, using two values for c, slightly different, but each quite approximate. The last equation will then, in neither case, be exactly satisfied, but, by a comparison of the errors, one will discover the value of c which makes the left member sensibly zero. By a similar interpolation between the values of the b, given severally by the first and second eliminations, we get the sensibly exact values of these quantities.

When it is proposed to neglect terms of the same order as m^5, the equation for c may be written

$$[-1][0][1] - \theta_1^2\,[[-1] + [1]] = 0;$$

or, when we substitute for the symbols their significations,

$$[(c^2 + 4 - \theta_0)^2 - 16c^2][c^2 - \theta_0] - 2\theta_1^2\,[c^2 + 4 - \theta_0] = 0.$$

But, as $c^2 - \Theta_0$ is a quantity of the third order, we may neglect the cube of it in the first term, and the product of it by Θ_1^2 in the second. Thus

reduced, the equation becomes

$$[c^2 - \theta_0]^2 + 2[\theta_0 - 1][c^2 - \theta_0] + \theta_1^2 = 0;$$

whose solution gives

$$c = \sqrt{1 + \sqrt{(\theta_0 - 1)^2 - \theta_1^2}}.$$

This is a remarkably simple expression for obtaining an approximate value of the motion of the lunar perigee. The actual numerical values of the two elements entering into this formula are

$$\theta_0 = 1.1588439, \quad \theta_1 = -0.0570440.$$

Θ_1 is therefore more than one third of $\Theta_0 - 1$, which explains why such an erroneous value is obtained for the motion of the lunar perigee, when we neglect it, and take $c = \sqrt{\Theta_0}$. The numbers being substituted in the formula, we get $c = 1.0715632$; and as the ratio of the motion of the perigee to the sidereal mean motion of the moon is given by the equation

$$\frac{1}{n}\frac{d\omega}{dt} = 1 - \frac{c}{1 + m},$$

we get

$$\frac{1}{n}\frac{d\omega}{dt} = 0.008591.$$

This is about $\frac{1}{60}$ in excess of the value 0.008452 given by observation. The difference is caused, in the main, by our neglect of the inclination of the lunar orbit. The solar force is less effective in producing motion in the perigee than it would be if the moon moved in the plane of the ecliptic.

It will occur immediately to every one that the properties we have stated of the roots of $\mathfrak{D}(c) = 0$ are precisely those of the transcendental equation

$$\cos(\pi x) - a = 0;$$

of which, if x_0 is one of the roots, the whole series of roots is represented by

$$\pm x_0 + 2i.$$

Hence we must necessarily have, identically,

$$\mathfrak{D}(c) = A[\cos(\pi c) - \cos(\pi c_0)],$$

A being some constant independent of c. As is the general custom, we assume that the positive sign is given to the element of the determinant formed by the product of the diagonal line of constituents containing c. When, therefore, the determinant $\mathfrak{D}(c)$ is developed in powers of c, using

only a finite number of constituents in it, the coefficient of the highest power of c in it is always positive unity; hence we may assume that this is the value of the coefficient when the number of constituents is increased without limit. But from the well-known equation

$$\cos(\pi c) = (1 - \tfrac{4}{1}c^2)(1 - \tfrac{4}{9}c^2)(1 - \tfrac{4}{25}c^2)\ldots,$$

we gather that the coefficient of the highest power of c, in the development of $\cos(\pi c)$ in powers of c, may be regarded as represented by the infinite product

$$-\tfrac{4}{1}\cdot -\tfrac{4}{9}\cdot -\tfrac{4}{25}\cdot\ldots.$$

If then the row of constituents of $\mathfrak{D}(c)$, containing $[0]$, is multiplied by -4, the rows containing $[-1]$ and $[1]$ by $\frac{4}{4^2-1}$, the rows containing $[-2]$ and $[2]$ by $\frac{4}{8^2-1}$, and, in general, the row containing $[i]$ by $\frac{4}{(4i)^2-1}$, we shall have the constituents of a second determinant, which may be designated as $\nabla(c)$. And the equation

$$\nabla(c) = 0,$$

having the same roots as $\mathfrak{D}(c) = 0$, will serve our purposes as well as the latter. We evidently now have

$$\nabla(c) = \cos(\pi c) - \cos(\pi c_0).$$

As this is an identical equation, it holds when any special value is attributed to c, and we are thus furnished with an elegant method of obtaining the value of the absolute term of the equation $\cos(\pi c_0)$. For example, substituting for c, in succession, the values 0, $\frac{1}{2}$, 1, $\sqrt{\Theta_0}$, we have our choice between the values

$$\begin{aligned}\cos(\pi c_0) &= 1 - \nabla(0)\\ &= -\nabla(\tfrac{1}{2})\\ &= -1 - \nabla(1)\\ &= \cos(\pi\sqrt{\theta_0}) - \nabla(\sqrt{\theta_0}).\end{aligned}$$

As the determinant $\nabla(0)$ appears the simplest, we retain the first expression. Then, dropping the now useless subscript $(_0)$, the equation which determines c may be written

$$\cos(\pi c) = 1 - \nabla(0).$$

This is certainly a remarkable equation: it virtually amounts to a general solution of the equation $\mathfrak{D}(c) = 0$. It also affords us immediately the

criterion for the reality of the roots of the latter. Using the phrase of Cauchy, if the modulus of the quantity $1-\nabla(0)$ does not exceed unity, the roots are all real; in the contrary case, they are all imaginary. The criterion for deciding whether the variable w is always contained between definite limits, or is capable of increasing or diminishing beyond every limit, is the same. In the first case, it is developable in a series of circular cosines; in the second, in a series of potential cosines.

As, in the particular case, where Θ_1, Θ_2, &c., all vanish, the proper value of c is $\sqrt{\Theta_0}$, it follows that the element of the determinant $\nabla(0)$, formed by the product of the diagonal line of constituents involving Θ_0, is

$$1-\cos(\pi\sqrt{\theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

If therefore each row of constituents of the determinant $\nabla(0)$ is divided by the constituent of it which lies in the just-mentioned diagonal line, we shall have a set of constituents forming a third determinant $\Box(0)$, such that

$$\nabla(0) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right).\Box(0).$$

In consequence the equation, determining c, can be put in the form

$$\frac{\sin^2\left(\frac{\pi}{2}c\right)}{\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)} = \Box(0).$$

For the sake of exhibiting more clearly the significance of this equation, I write a few of the central constituents of the determinant $\Box(0)$, from which the rest can be easily inferred.

$$\Box(0) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots + & 1 & -\frac{\theta_1}{4^2-\theta_0} & -\frac{\theta_2}{4^2-\theta_0} & -\frac{\theta_3}{4^2-\theta_0} & -\frac{\theta_4}{4^2-\theta_0} & \cdots \\ \cdots -\frac{\theta_1}{2^2-\theta_0} & + & 1 & -\frac{\theta_1}{2^2-\theta_0} & -\frac{\theta_2}{2^2-\theta_0} & -\frac{\theta_3}{2^2-\theta_0} & \cdots \\ \cdots -\frac{\theta_2}{0^2-\theta_0} & -\frac{\theta_1}{0^2-\theta_0} & + & 1 & -\frac{\theta_1}{0^2-\theta_0} & -\frac{\theta_2}{0^2-\theta_0} & \cdots \\ \cdots -\frac{\theta_3}{2^2-\theta_0} & -\frac{\theta_2}{2^2-\theta_0} & -\frac{\theta_1}{2^2-\theta_0} & + & 1 & -\frac{\theta_1}{2^2-\theta_0} & \cdots \\ \cdots -\frac{\theta_4}{4^2-\theta_0} & -\frac{\theta_3}{4^2-\theta_0} & -\frac{\theta_2}{4^2-\theta_0} & -\frac{\theta_1}{4^2-\theta_0} & + & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The question of the convergence, so to speak, of a determinant, consisting of an infinite number of constituents, has nowhere, so far as I am aware,

been discussed. All such determinants must be regarded as having a central constituent; when, in computing in succession the determinants formed from the 3^2, 5^2, 7^2, &c., constituents symmetrically situated with respect to the central constituent, we approach, without limit, a determinate magnitude, the determinant may be called convergent, and the determinate magnitude is its value.

In the present case, there can scarcely be a doubt that, as long as the series $\Sigma_i \,.\, \Theta_i \zeta^{2i}$ is a legitimate expansion of Θ, the determinant $\square(0)$ must be regarded as convergent.

We will give another equation for determining c. We have

$$\cos(\pi c) = \cos(\pi\sqrt{\theta_0}) - \nabla(\sqrt{\theta_0}).$$

The diagonal line of constituents in $\nabla(\sqrt{\Theta_0})$ is represented in general by the formula

$$\frac{16i\,(i+\sqrt{\theta_0})}{(4i)^2-1};$$

and when the factor corresponding to $i=0$ is omitted, the product

$$\prod_{i=-\infty}^{i=+\infty} \frac{16i\,(i+\sqrt{\theta_0})}{(4i)^2-1} = \frac{\pi\sin(\pi\sqrt{\theta_0})}{8\sqrt{\theta_0}}.$$

Consequently, if we put

$$\square(\sqrt{\theta_0}) = \left|\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots + 1 & -\frac{\theta_1}{8(2-\sqrt{\theta_0})} & -\frac{\theta_2}{8(2-\sqrt{\theta_0})} & -\frac{\theta_3}{8(2-\sqrt{\theta_0})} & -\frac{\theta_4}{8(2-\sqrt{\theta_0})} & \cdots \\
\cdots -\frac{\theta_1}{4(1-\sqrt{\theta_0})} & +1 & -\frac{\theta_1}{4(1-\sqrt{\theta_0})} & -\frac{\theta_2}{4(1-\sqrt{\theta_0})} & -\frac{\theta_3}{4(1-\sqrt{\theta_0})} & \cdots \\
\cdots + \theta_2 & +\theta_1 & +0 & +\theta_1 & +\theta_2 & \cdots \\
\cdots -\frac{\theta_3}{4(1+\sqrt{\theta_0})} & -\frac{\theta_2}{4(1+\sqrt{\theta_0})} & -\frac{\theta_1}{4(1+\sqrt{\theta_0})} & +1 & -\frac{\theta_1}{4(1+\sqrt{\theta_0})} & \cdots \\
\cdots -\frac{\theta_4}{8(2+\sqrt{\theta_0})} & -\frac{\theta_3}{8(2+\sqrt{\theta_0})} & -\frac{\theta_2}{8(2+\sqrt{\theta_0})} & -\frac{\theta_1}{8(2+\sqrt{\theta_0})} & +1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|,$$

a determinant which, having 0 for its central constituent, presents some facilities in its computation, we shall have, for determining c, the equation

$$\frac{\sin^2\left(\frac{\pi}{2}c\right)}{\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)} = 1 + \frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{2\sqrt{\theta_0}}\square(\sqrt{\theta_0}).$$

In the lunar theory Θ_i is a quantity of the $2i^{\text{th}}$ order, and $1-\sqrt{\Theta_0}$ a quantity of the first order; hence it is clear that, if we are willing to admit

an error of the seventh order in c, the determinant

$$\square(\sqrt{\theta_0}) = -\tfrac{1}{2}\frac{\theta_1^2}{\theta_0 - 1}.$$

If, neglecting, then, quantities of the seventh order, we put

$$\frac{\pi\theta_1^2}{8\sqrt{\theta_0}(\theta_0 - 1)} = \tan\theta,$$

θ will be a small angle, and c will result from the equation

$$\sin\left(\frac{\pi}{2}\,c\right) = \frac{\sin\left(\frac{\pi}{2}\sqrt{\theta_0} - \theta\right)}{\cos\theta}.$$

This formula, although it involves the same coefficients Θ_0 and Θ_1 as the approximate formula previously given, is two orders more exact. A greater degree of approximation can be arrived at only by including the additional coefficient Θ_2. Employing the numerical values already attributed to Θ_0 and Θ_1, we find

$$\theta = 25'\,41.''395, \qquad c = 1.0715815, \qquad \frac{1}{n}\frac{d\omega}{dt} = 0.008574.*$$

The determinants $\square(0)$ and $\square(\sqrt{\Theta_0})$ can be replaced by infinite series proceeding according to ascending powers and products of the coefficients Θ_1, Θ_2, &c.

Let us take the first, as being in more respects the simpler. It is plain that the element of the determinant formed by the product of the diagonal line of constituents is the only term of the zero order in it. Then one exchange always produces terms of the 4th or higher orders, two exchanges terms of the 8th or higher orders, three exchanges terms of the 12th or higher orders, and so on. Now let i, i', i'' .... be positive or negative integers, of which no two are identical, written in the order of their algebraical magnitude, and let $\{i\}$ stand for $(2i)^2 - \Theta_0$. Then all the terms of $\square(0)$, which

---

* It is better, however, to employ the equations

$$\frac{\pi\Theta_1}{4\sqrt{\Theta_0}(\Theta_0 - 1)} = \tan\theta, \quad \cos(\pi c) = \frac{\cos(\pi\sqrt{\Theta_0} - \theta)}{\cos\theta},$$

which give

$$\theta = 51'22''.6185, \quad c = 1.0715837865, \quad \frac{1}{n}\frac{d\omega}{dt} = 0.0085721020.$$

(Note of 1886.)

are obtained by 0, 1, 2, and 3 exchanges, are contained in the following expression, which is, consequently, affected with an error of the 16th order.

$$
\begin{aligned}
\square(0) = 1 &- \Sigma_{i,\,i'} \frac{\theta^2_{i'-i}}{\{i\}\{i'\}} \\
&+ \Sigma_{i,\,i',\,i'',\,i'''} \frac{\theta^2_{i'-i}\theta^2_{i'''-i''}}{\{i\}\{i'\}\{i''\}\{i'''\}} \\
&- 2\Sigma_{i,\,i',\,i''} \frac{\theta_{i'-i}\theta_{i''-i'}\theta_{i''-i}}{\{i\}\{i'\}\{i''\}} \\
&- \Sigma_{i,\,i',\,i'',\,i''',\,i^{\text{iv}},\,i^{\text{v}}} \frac{\theta^2_{i'-i}\theta^2_{i'''-i''}\theta^2_{i^{\text{v}}-i^{\text{iv}}}}{\{i\}\{i'\}\{i''\}\{i'''\}\{i^{\text{iv}}\}\{i^{\text{v}}\}} \\
&+ 2\Sigma_{i,\,i',\,i'',\,i''',\,i^{\text{iv}},} \frac{\left\{\begin{array}{l} \theta_{i'-i}\theta^2_{i'''-i''}\theta_{i^{\text{iv}}-i}\theta_{i^{\text{iv}}-i'} \\ +\, \theta^2_{i'-i}\theta_{i'''-i''}\theta_{i^{\text{iv}}-i''}\theta_{i^{\text{iv}}-i'''} \end{array}\right\}}{\{i\}\{i'\}\{i''\}\{i'''\}\{i^{\text{iv}}\}} \\
&- 2\Sigma_{i,\,i',\,i'',\,i'''} \frac{\left\{\begin{array}{l} \theta_{i'-i}\theta_{i''-i'}\theta_{i'''-i''}\theta_{i'''-i} \\ +\, \theta_{i''-i}\theta_{i''-i'}\theta_{i'''-i'}\theta_{i'''-i} \\ +\, \theta_{i'-i}\theta_{i''-i}\theta_{i'''-i'}\theta_{i'''-i''} \end{array}\right\}}{\{i\}\{i'\}\{i''\}\{i'''\}}.
\end{aligned}
$$

Particularizing the summations in this expression, and retaining only terms which are of lower orders than the 16th, we get

$$
\left.
\begin{aligned}
\square(0) = 1 &- \theta_1^2\Sigma_i \frac{1}{\{i\}\{i+1\}} - \theta_2^2\Sigma_i \frac{1}{\{i\}\{i+2\}} - \theta_3^2\Sigma_i \frac{1}{\{i\}\{i+3\}} \\
&+ \theta_1^4\Sigma_{i,\,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}} \\
&+ \theta_1^2\theta_2^2\Sigma_{i,\,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+2\}} \\
&- 2\theta_1^2\theta_2\Sigma_i \frac{1}{\{i\}\{i+1\}\{i+2\}} \\
&- 2\theta_1\theta_2\theta_3\Sigma_i \frac{1}{\{i\}\{i+1\}\{i+3\}} \\
&- 2\theta_1\theta_2\theta_3\Sigma_i \frac{1}{\{i\}\{i+2\}\{i+3\}} \\
&- \theta_1^6\Sigma_{i,\,i',\,i''} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i''\}\{i''+1\}} \\
&+ 2\theta_1^4\theta_2\Sigma_{i,\,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i'+2\}} \\
&- 2\left[\theta_1^2\theta_2^2 + \theta_1^3\theta_3\right]\Sigma_i \frac{1}{\{i\}\{i+1\}\{i+2\}\{i+3\}}.
\end{aligned}
\right\} \quad (23)
$$

The functions of Θ_0, which are represented by the summations, can all be replaced by finite expressions. For brevity, let us put $\Theta_0 = 4\theta^2$, then

resolving the expression into partial fractions, i being taken as the variable, we have, for instance,

$$\Sigma_i \frac{1}{\{i\}\{i+k\}} = \tfrac{1}{16}\Sigma_i \frac{1}{(\theta+i)(\theta-i)(\theta+i+k)(\theta-i-k)}$$
$$= \tfrac{1}{16}\Sigma_i \left[\frac{A}{\theta+i} + \frac{B}{\theta-i} + \frac{C}{\theta+i+k} + \frac{D}{\theta-i-k}\right],$$

where A, B, C, and D are determined by the equations

$$2k\theta(2\theta-k)A = 1,$$
$$-2k\theta(2\theta+k)B = 1,$$
$$-2k\theta(2\theta+k)C = 1,$$
$$2k\theta(2\theta-k)D = 1.$$

But, as is well known,

$$\Sigma_i \frac{1}{\theta+i} = \Sigma_i \frac{1}{\theta-i} = \Sigma_i \frac{1}{\theta+i+k} = \Sigma_i \frac{1}{\theta-i-k} = \pi \cot \pi\theta .$$

Consequently,

$$\Sigma_i . \frac{1}{\{i\}\{i+k\}} = \tfrac{1}{16}(A+B+C+D)\,\pi\cot\pi\theta$$
$$= \frac{\pi\cot\pi\theta}{8\theta(4\theta^2-k^2)}$$
$$= \frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0-k^2)}.$$

In like manner will be found

$$\Sigma_i \frac{1}{\{i\}\{i+k\}\{i+k'\}} = -\tfrac{1}{16}\frac{3\theta_0-(k^2-kk'+k'^2)}{\sqrt{\theta_0}(\theta_0-k^2)(\theta_0-k'^2)[\theta_0-(k-k')^2]}\,\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right),$$

$$\Sigma_i \frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}$$
$$= -\tfrac{1}{32}\frac{5\theta_0-(k^2+1)}{\sqrt{\theta_0}(\theta_0-1)(\theta_0-k^2)[\theta_0-(k+1)^2][\theta_0-(k-1)^2]}\,\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

By attributing, in these equations, special integral values to k, will be obtained the values of all the single summations appearing in the preceding expression for $\square(0)$. With regard to the double summations, we may proceed as follows: Substitute $i+k$ for i', then resolve the expression under consideration into partial fractions with respect to i as variable, and sum between the limits $-\infty$ and $+\infty$; the fractions occurring in the result thus obtained are next resolved into partial fractions with reference to k, and the summations, with reference to this integer, are taken between the limits 2 and $+\infty$; or, which is the same thing, between the limits 0 and $+\infty$.

and the terms corresponding to $k=0$ and $k=1$ subtracted from the result, The single triple summation may be treated in an analogous manner. Thus we get

$$\Sigma_{i,i'}\frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}}=\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right],$$

$$\Sigma_{i,i'}\frac{1}{\{i\}\{i+1\}\{i'\}\{i'+2\}}$$
$$=\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{5}{9-\theta_0}\right],$$

$$\Sigma_{i,i'}\frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i'+2\}}$$
$$=\frac{3\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{64\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{20}{3(9-\theta_0)}\right],$$

$$\Sigma_{i,i',i''}\frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i''\}\{i''+1\}}$$
$$=-\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\left[-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{25}{8\theta_0}\right.$$
$$\left.+\frac{1}{\theta_0^2}+\frac{2}{1-\theta_0}+\frac{4}{(1-\theta_0)^2}-\frac{9}{8(4-\theta_0)}+\frac{9}{(4-\theta_0)^2}-\frac{4}{9-\theta_0}-\frac{\pi^2}{3\theta_0}\right\}.$$

From which it follows that

$$
\left.\begin{aligned}
\square(0)=1&+\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}}\left[\frac{\theta_1^2}{1-\theta_0}+\frac{\theta_2^2}{4-\theta_0}+\frac{\theta_3^2}{9-\theta_0}\right]\\
&+\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\theta_1^4\\
&+\frac{3\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{8\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\theta_1^2\theta_2\\
&+\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\left[-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{25}{8\theta_0}-\frac{1}{\theta_0^2}\right.\\
&\left.+\frac{2}{1-\theta_0}+\frac{4}{(1-\theta_0)^2}-\frac{9}{8(4-\theta_0)}+\frac{9}{(4-\theta_0)^2}-\frac{4}{9-\theta_0}-\frac{\pi^2}{3\theta_0}\right\}\theta_1^6\\
&+\frac{3\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{20}{3(9-\theta_0)}\right]\theta_1^4\theta_2\\
&+\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{10}{9-\theta_0}\right]\theta_1^2\theta_2^2\\
&+\frac{(7-3\theta_0)\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1\theta_2\theta_3+\frac{5\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1^3\theta_2.
\end{aligned}\right\}\quad(24)
$$

This is the same result as would be obtained if, setting out with the equation $\mathfrak{D}(\mathrm{c}) = 0$, and assuming that $\mathrm{c} = \sqrt{\Theta_0}$ is an approximate value, we should expand the function $\sin^2\left(\frac{\pi}{2}\,\mathrm{c}\right)$ in ascending powers and products of the coefficients Θ_1, Θ_2, &c.

IV.

In order to obtain a numerical result from the preceding investigation, we assume

$$n = 17325594''.06085, \qquad n' = 1295977''.41516,$$

whence

$$\mathrm{m} = 0.08084\ 89338\ 08311.6.$$

From an investigation (to be published hereafter) of the corresponding values of the a_i, we have

$$\begin{array}{rlrl}
2h_1 = & +\ 0.00909\ 42448\ 77375.5 & -\ 2h_{-1} = & -\ 0.01739\ 14939\ 23079.4 \\
4h_2 = & +\ 0.00011\ 75731\ 31569.1 & -\ 4h_{-2} = & +\ 0.00000\ 19654\ 85829.2 \\
6h_3 = & +\ 0.00000\ 12613\ 28523.8 & -\ 6h_{-3} = & +\ 0.00000\ 00738\ 11780.8 \\
8h_4 = & +\ 0.00000\ 00126\ 19314.9 & -\ 8h_{-4} = & +\ 0.00000\ 00006\ 87885.7 \\
10h_5 = & +\ 0.00000\ 00001\ 21722.9 & -10h_{-6} = & +\ 0.00000\ 00000\ 05777.1 \\
12h_6 = & +\ 0.00000\ 00000\ 01147.9 & -12h_{-6} = & +\ 0.00000\ 00000\ 00047.5 \\
14h_7 = & +\ 0.00000\ 00000\ 00010.6 & -14h_{-7} = & +\ 0.00000\ 00000\ 00000.4
\end{array}$$

The values of the U_i derived from these are

$$\begin{array}{rlrl}
U_1 = & +\ 0.00909\ 40932\ 76038.2 & U_{-1} = & -\ 0\ 01739\ 21860\ 78260.6 \\
U_2 = & +\ 0.00007\ 62192\ 02104.5 & U_{-2} = & +\ 0.00015\ 32094\ 08075.6 \\
U_3 = & +\ 0.00000\ 06474\ 24628.8 & U_{-3} = & -\ 0.00000\ 12670\ 56302.6 \\
U_4 = & +\ 0.00000\ 00055\ 23086.8 & U_{-4} = & +\ 0.00000\ 00115\ 67648.9 \\
U_5 = & +\ 0.00000\ 00000\ 47209.0 & U_{-6} = & -\ 0.00000\ 00000\ 95049.5 \\
U_6 = & +\ 0.00000\ 00000\ 00403.9 & U_{-6} = & +\ 0.00000\ 00000\ 00867.3 \\
U_7 = & +\ 0.00000\ 00000\ 00003.4 & U_{-7} = & -\ 0.00000\ 00000\ 00007.2
\end{array}$$

In combination with the values of R_i, which will be given elsewhere, these afford the following periodic series for Θ:

$$\begin{aligned}
\theta = \ & 1.15884\ 39395\ 96583 \\
& -\ 0.11408\ 80374\ 93807 \cos 2\tau \\
& +\ 0.00076\ 64759\ 95109 \cos 4\tau \\
& -\ 0.00001\ 83465\ 77790 \cos 6\tau \\
& +\ 0.00000\ 01088\ 95009 \cos 8\tau \\
& -\ 0.00000\ 00020\ 98671 \cos 10\tau \\
& +\ 0.00000\ 00000\ 12103 \cos 12\tau \\
& -\ 0.00000\ 00000\ 00211 \cos 14\tau
\end{aligned}$$

The values of the coefficients Θ_0, Θ_1, Θ_2, &c., are the halves of these coefficients, except Θ_0, which is equal to the first coefficient.

On substituting the numerical values of these quantities in (24), and separating the sum of the terms into groups according to their order, for the sake of exhibiting the degree of convergence, we get

Term of the zero order,	1.00000 00000 00000 0
Term of the 4th order,	+ 0.00180 46110 93422 7
Sum of the terms of the 8th order,	+ 0.00000 01808 63109 9
Sum of the terms of the 12th order,	+ 0.00000 00000 64478 6
$\Box(0) =$	1.00180 47920 21011 2

As far as we can judge from induction, the value of $\Box(0)$ would be affected, only in the 14th decimal, by the neglected remainder of the series, which is of the 16th order. An error in $\Box(0)$ is multiplied by 2.8 nearly in c.

The value, which is derived thence for c, is

$$c = 1.07158\ 32774\ 16016.$$

In order that nothing may be wanting in the exact determination of this quantity, we will employ the value just obtained as an approximate value in the elimination between equations (21). The coefficients $[i]$, as many of them as we have need for, have the following values:

$$\begin{aligned} [-4] &= 46.8, & [1] &= 8.27577\ 98905\ 1, \\ [-3] &= 23.13045, & [2] &= 24.56211\ 3, \\ [-2] &= 7.41678\ 05615\ 1, & [3] &= 48.85. \\ [-1] &= -0.29688\ 63288\ 2300, \end{aligned}$$

If the quantities b_i are eliminated from equations (21) in the order b_{-1}, b_1, b_{-2}, b_2, b_{-3}, b_3, and b_{-4}, it will be found that the coefficient of b_0, in the principal equation, undergoes the following successive depressions:

$$\begin{aligned} &[0] &&= -0.01055\ 32191\ 58933, \\ &[0]^{(-1)} &&= +0.00040\ 72723\ 11650, \\ &[0]^{(-1,1)} &&= +0.00001\ 50888\ 08423, \\ &[0]^{(-2,-1,1)} &&= +0.00000\ 00253\ 21700, \\ &[0]^{(-2,-1,1,2)} &&= +0.00000\ 00009\ 20420, \\ &[0]^{(-3,-2,-1,1,2)} &&= +0.00000\ 00000\ 03941, \\ &[0]^{(-3,-2,-1,1,2,3)} &&= +0.00000\ 00000\ 00155, \\ &[0]^{(-4,-3,-2,-1,1,2,3)} &&= +0.00000\ 00000\ 00008. \end{aligned}$$

The last number is not sensibly changed by the elimination of any of the b_i beyond b_{-4} on the one side, or b_3 on the other. This residual is so small that it will not be necessary to repeat the computation with another value of c : it will suffice to subtract half of it from the assumed value of c. Thus we have as the final result:

$$c = 1.07158\,32774\,16012;$$

and, consequently,

$$\frac{1}{n}\frac{d\omega}{dt} = 0.00857\,25730\,04864.$$

Let us compare this value with that obtained from Delaunay's literal expression,*

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{4}m^2 + \frac{225}{32}m^3 + \frac{4071}{128}m^4 + \frac{265493}{2048}m^5 + \frac{12822631}{24576}m^6$$
$$+ \frac{1273925965}{589824}m^7 + \frac{71028685589}{7077888}m^8 + \frac{32145882707741}{679477248}m^9,$$

where m denotes the ratio of the mean motions of the sun and moon. On the substitution of the numerical values we have employed for these quantities, this series gives, term by term,

$$\frac{1}{n}\frac{d\omega}{dt} = 0.00419\,6429 + 0.00294\,2798 + 0.00099\,5700 + 0.00030\,3577$$
$$+ 0.00009\,1395 + 0.00002\,8300 + 0.00000\,9836 + 0.00000\,3468 = 0.00857\,1503.$$

From the comparison, it appears that the sum of the remainder of Delaunay's series is 0.00000 1070, somewhat less than would be inferred by induction from the terms of the series itself. And, although Delaunay has been at the great pains of computing 8 terms of this series, they do not suffice to give correctly the first 4 significant figures of the quantity sought. On the other hand, the terms of the highest order, computed in the expression for $\square(0)$, were of the 12^{th} order only; and yet, as we have seen, they have sufficed for giving c exact nearly to the 15^{th} decimal. As well as can be judged from induction, it would be necessary to prolong the series, in powers of m, as far as m^{27}, in order to obtain an equally precise result. Allowing that the two last figures of the foregoing value of $\frac{1}{n}\frac{d\omega}{dt}$ may be vitiated by the accumulation of error arising from the very numerous operations, we may, I think, assert that 13 decimals correctly correspond to the assumed value of m. It may be stated that all the computations have been made twice, and no inconsiderable portion of them three times.

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* *Comptes Rendus de l'Académie des Sciences de Paris*, Tom. LXXIV, p. 19.

MEMOIR No. 30.

Empirical Formula for the Volume of Atmospheric Air.

(Analyst, Vol. IV, pp. 97-107, 1877.)

The formula of Mariotte and Gay-Lussac is generally employed, in the laboratory, to reduce volumes, observed under one tension and temperature, to those which would have place under other tensions and temperatures. But Regnault, about 1845, made several series of experiments, which, if they may be relied upon, establish marked deviations from this formula. These experiments are detailed in the *Mémoires de l'Académie des Sciences de Paris*, Tom. XXI. I propose to investigate a modification of the formula, the introduction of which makes it possible to satisfy nearly these experiments.

Let T denote the temperature, here always expressed in degrees of the centigrade scale; P the tension or pressure, measured by the altitude, in meters, of a column of mercury, it is capable of supporting, the mercury being at the temperature 0^0 and under the action of gravity which obtains at Regnault's laboratory; and let V denote the volume. Then, for any given mass of air, these three quantities are so connected that, if any two of them are assigned, the remaining third is immediately determined. That is, we must have

$$\text{function}\,(V, P, T) = 0,$$

or, solved with respect to V,

$$V = \text{function}\,(P, T).$$

But the mode, in which T is to be measured, is arbitrary, and we may take atmospheric air as the thermometric substance, and assume that T increases, in direct proportion, as the volume, under constant pressure, increases. This gives

$$V = F(P) + f(P) \cdot T.$$

It is here taken for granted that, whatever may be the density of the air inclosed in the thermometer, its indications will be the same. It is true that the usual custom of experimenters has been to measure temperatures

by the augmentation of tensions under constant volume; but, when Mariotte's law holds, this gives results identical with those obtained by the former method. In this case we should have to write the equation

$$P = F(V) + f(V) \,.\, T,$$

but the first equation seems preferable.

Now since, for any given constant temperature, the volume ought to be a function of the tension similar to what it is at any other temperature, it follows that, if $F(P)$ is supposed to consist of a series of terms, each of the form KP^k, where K and k are constants, so that we may write

$$F(P) = \Sigma \,.\, KP^k,$$

then we ought to have

$$f(P) = \Sigma \,.\, K_1 P^k,$$

where K_1 denotes a constant, in general, different from K. Thus we should have

$$V = \Sigma \,.\, [K + K_1 T]\, P^k.$$

The formula of Mariotte and Gay-Lussac assumes that $F(P)$ and $f(P)$ contain each only one term, in which $k = -1$. But Regnault's experiments having shown the insufficiency of this, it is in order to see whether agreement between theory and observation cannot be brought about by annexing to V an additional term, for which k has a value different from -1. Thus let us suppose that

$$\begin{aligned} V &= [K + K_1 T]\, P^{-1} + [K' + K_1' T]\, P^{\beta-1} \\ &= \frac{K + K'P^\beta}{P} + \frac{K_1 + K_1'P^\beta}{P}\, T. \end{aligned}$$

As V contains a factor, which is directly proportional to the mass of air considered, and inversely as the unit assumed for the measurement of volumes, we prefer to write the preceding equations thus:

$$\begin{aligned} V &= K\left[\frac{1 + \alpha' T}{P} + \frac{\alpha + \alpha'' T}{P}\, P^\beta\right] \\ &= K\left[\frac{1 + \alpha P^\beta}{P} + \frac{\alpha' + \alpha'' P^\beta}{P}\, T\right]. \end{aligned}$$

When the temperature is constant, the volumes are represented by the formula

$$V = K\,\frac{1 + \alpha P^\beta}{P},$$

that is, the result from Mariotte's law must be multiplied by the factor $1 + \alpha P^{\beta}$, which differs but little from unity; α is a small constant which measures the amplitude of the deviations from this law; while β is a constant exponent so chosen that the more or less rapid variation of the deviations, in passing from one tension to another, may be represented as well as possible. It is evident that, in this manner, we get the utmost advantage that can be derived from the addition of a single term to V.

The experiments of Regnault may be divided into two classes; first, those where, the temperature remaining nearly constant, the volumes of the same mass of air, under different pressures, were observed; second, those where, the volumes remaining nearly the same, the tensions were observed at the temperature of freezing and boiling water. It is obvious that experiments of these two kinds, extended over a sufficient range of tension, would afford the data requisite for obtaining the values of the four constants α, α', α'' and β which enter into our adopted formula.

The experiments of the first class are enumerated at pp. 374–379 of the volume quoted above. As the temperature is nearly the same for all, we assume that they have been made at the average of all the noted temperatures which is 4°.747.

To save labor, we may take the average of the observed volumes and tensions when they are nearly alike. In this way Regnault's 66 experiments are reduced to the 23 given in the following table. It may be noted that V is here expressed by the number of grammes of mercury required to fill the volume. The column containing $\log (PV)$ exhibits the deviation from Mariotte's law; did this law exactly hold, the numbers in this column would be identical for each series. It will be noted that, in general, they diminish with increasing pressures. The volumes being supposed to be represented by the equation

$$V = K\frac{1 + \alpha P^{\beta}}{P},$$

a preliminary investigation has given the approximate values

$$\alpha = -0.0024337, \quad \beta = 0.645.$$

With these have been computed the values of the expressions which stand at the head of the two last columns of the table, and which serve to obtain the coefficients of the equations of condition to be given presently.

As the mass of air operated on was different in each series of experiments, K will have 9 different values; it can, however, be eliminated.

Taking the common logarithms of each member of the equation last given,

$$\log K + \log(1 + aP^{\beta}) = \log(PV).$$

Series.	$V.$	$P.$	No. Obs.	$\log(PV).$	$\frac{P^{\beta}}{1 + aP^{\beta}}.$	$\frac{P^{\beta}}{1 + aP^{\beta}} \log P.$
I.	1939.76	0.73899	4	3.156387	0.8244	—0.1083
	969.65	1.47630	4	3.155790	1.2897	+0.2182
II.	1939.37	2.11228	3	3.612412	1.6262	0.5281
	970.40	4.21020	3	3.611254	2.5430	1.5876
	642.82	6.35034	2	3.610886	3.3213	2.6664
III.	1939.72	2.06887	3	3.603472	1.6045	0.5066
	969.78	4.12663	6	3.602268	2.5102	1.5452
IV.	1940.65	4.14235	2	3.905194	2.5164	1.5532
	979.78	8.17850	3	3.903803	3.9155	3.5737
V.	1939.85	4.21910	4	3.912988	2.5465	1.5921
	970.29	8.40648	4	3.911516	3.9863	3.6857
	626.91	12.98195	1	3.910545	5.2926	5.8925
VI.	1940.23	6.77001	3	4.118444	3.4623	2.8758
	970.32	13.47353	4	4.116396	5.4226	6.1247
	685.11	19.00213	1	4.114562	6.7913	8.6846
	675.15	19.30191	2	4.115000	6.8612	8.8206
VII.	1941.23	6.39003	2	4.093580	3.3347	2.6861
	969.98	12.72859	2	4.091543	5.2248	5.7721
	633.82	19.39954	1	4.089757	6.8842	8.8654
VIII.	1940.44	9.33401	3	4.257968	4.2676	4.1398
	970.53	18.54702	5	4.255283	6.6842	8.4774
IX.	1945.06	11.47357	2	4.348632	4.8824	5.1740
	1053.78	21.05700	2	4.346146	7.2643	9.6137

To reduce the matter within the treatment of the method of least squares, it will be necessary to make some assumption regarding the probable errors of the observed P and V. We will, for convenience, suppose that they are such that the function $\log(PV)$ has a probable error equal for all the observations; an assumption somewhat precarious, it is true, but it seems that we cannot easily do better.

Let the small corrections, which it is necessary to apply to the approximate values of $\log K$, α and β, be denoted by $\delta \log K$, $\delta\alpha$ and $\delta\beta$, and let us put

$$\delta \log K = x, \quad M\delta a = y, \quad a\delta\beta = z,$$

where M denotes the modulus of common logarithms. $\delta \log(PV)$ being the excess of observed over calculated $\log(PV)$, we shall have the equation of condition:

$$x + \frac{P^{\beta}}{1 + aP^{\beta}} y + \frac{P^{\beta}}{1 + aP^{\beta}} \log P \,.\, z = \delta(PV).$$

A little consideration will show that x will be eliminated by taking the difference of every two equations of condition arising from the same series, and attributing the weight $\frac{ww'}{\Sigma . w}$ to the resulting equation, w and w' denoting the weights of the equations whose difference is taken, and $\Sigma . w$ the sum of the weights of all the equations in the series. Since the coefficients of y and z, in the equations, are all positive and nearly proportional, it will be advantageous to adopt a new unknown u, such that

$$y = u - \tfrac{5}{8} z.$$

Then the equations, with the weights that ought to be attributed to them, are

Series.				Weight.
I.	$0.4653u$	$-0.4490z$	$= -0.000106$	2
II.	0.9168	-0.4685	$= -0.000193$	$\frac{9}{8}$
	1.6951	-0.6869	$= +0.000255$	¾
	0.7783	-0.2184	$= +0.000448$	¾
III.	0.9057	-0.4709	$= -0.000252$	2
IV.	1.3991	-0.3113	$= +0.000076$	1.2
V.	1.4398	-0.3061	$= +0.000038$	$\frac{16}{9}$
	2.7461	-0.2764	$= +0.000432$	$\frac{4}{9}$
	1.3063	$+0.0296$	$= +0.000394$	$\frac{4}{9}$
VI.	1.9603	-0.0183	$= +0.000002$	1.2
	3.3290	$+0.2605$	$= -0.000407$	0.3
	3.3989	$+0.2800$	$= +0.000104$	0.6
	1.3687	$+0.2787$	$= -0.000409$	0.4
	1.4386	$+0.2982$	$= +0.000102$	0.8
	0.0699	$+0.0195$	$= +0.000511$	0.2
VII.	1.8901	-0.0642	$= -0.000059$	0.8
	3.5495	$+0.2635$	$= -0.000117$	0.4
	1.6594	$+0.3276$	$= -0.000058$	0.4
VIII.	2.4166	$+0.3099$	$= -0.000164$	$\frac{15}{8}$
IX.	2.3819	$+0.4699$	$= -0.000005$	1

The derived normal equations are

$$58.672\,u - 0.0790\,z = -\,0.0005252,$$
$$-\,0.079\,u + 2.4453\,z = -\,0.0000157.$$

Whence

$u = -\,0.000008962$, $z = -\,0.000006707$, $y = +\,0.000002216$,
$\delta\alpha = +\,0.0000051$, $\delta\beta = +\,0.00276$.

Applying these corrections to the approximate values of α and β, we get

$$\alpha = -\,0.0024286, \quad \beta = +\,0.64776.$$

How well the experiments are represented by the formula, with these values of the constants, will best be seen from the following comparison of the values of $\frac{V_0 P_0}{V_1 P_1}$ given by Regnault and those computed from the formula:

Obs.	Cal.	Diff.	Obs.	Cal.	Diff.
1.001414	1.001133	+281	1.005437	1.006694	—1257
1.001448	1.001132	+316	1.005703	1.006694	— 991
1.001224	1.001133	+ 91			
1.001421	1.001133	+288	1.004286	1.004777	— 491
			1.004512	1.004770	— 258
1.002765	1.002233	+532	1.004599	1.004779	— 180
1.002759	1.002234	+525	1.004580	1.004771	— 191
1.002503	1.002236	+267	1.008536	1.008106	+ 430
1.003539	1.004134	—595	1.008813	1.008108	+ 705
1.003452	1.004133	—681	1.008016	1.008286	— 270
1.003309	1.004133	—824	1.008064	1.008269	— 205
			1.007980	1.008288	— 308
1.002709	1.002209	+500			
1.002724	1.002207	+517	1.004611	1.004601	+ 10
1.002713	1.002206	+507	1.004752	1.004601	+ 151
1.002528	1.002211	+317	1.008930	1.008648	+ 282
1.002898	1.002203	+695	1.008755	1.008642	+ 113
1.002762	1.002203	+559			
			1.006366	1.005876	+ 490
1.003253	1.003417	—164	1.006132	1.005880	+ 252
1.003090	1.003411	—321	1.006010	1.005869	+ 141
1.003302	1.003407	—105	1.006346	1.005878	+ 468
1.003336	1.003506	—170	1.005619	1.005738	— 121
1.003495	1.003508	— 13	1.005622	1.005736	— 114
1.003335	1.003508	—173	1.005902	1.005832	+ 70
1.003448	1.003509	— 61			

It will be seen that the differences, in the extreme cases, amount to a fourth part of the observed deviation from the law of Mariotte. Moreover, it is plain that some cause, which, varied from series to series, has operated to vitiate these experiments, since it is possible to determine α and β so that any two series are well represented, but not possible when all the series are included in the investigation. It may be noted also that the experiments, in which the original volume was reduced to one-third, are not, in general, concordant with those where the reduction was to one-half.

That these discrepancies are unavoidable will be evident from the following exposition: Let us put

$$\text{com. log}(PV) = F(P).$$

The observations of Regnault may be condensed into the following nine results, all formed by combining tolerably concordant data:

1.	$F(1.476)-F(0.739)=0.000598$
2.	$F(4.168)-F(2.091)=0.001181$
3.	$F(6.350)-F(2.112)=0.001526$
4.	$F(8.292)-F(4.182)=0.001437$
5.	$F(12.982)-F(4.219)=0.002443$
6.	$F(13.101)-F(6.580)=0.002042$
7.	$F(19.276)-F(6.580)=0.003743$
8.	$F(18.547)-F(9.334)=0.002685$
9.	$F(21.057)-F(11.474)=0.002486$

These are the data actually furnished by Regnault for the determination of the function $F(P)$. Employing the graphical method, we endeavor to construct the curve whose equation is $y=F(x)$. One of the special values of $F(x)$, as $F(0.739)$, may be taken arbitrarily, and then the value of $F(1.476)$ becomes known. This premised, we see that each of the nine equations furnishes the length, direction and abscissæ of the extremities of a chord, of the sought curve. Placing the chord, corresponding to the first equation, arbitrarily, and drawing the others on any part of the paper, but with the proper direction and abscissæ of their extremities, we endeavor, by imparting a motion to all their points parallel to the axis of y, to make them fall into line as chords of the same continuous curve. We find that if we take 1, 2, 4, 6 and 7, they can be made to indicate a tolerably continuous curve; but then 3, 5, 8 and 9 are not satisfied.

Again, from this graphical process, we see that there cannot be much variation of curvature between the extremities of each chord, and hence the tangent to the curve, corresponding to the abscissa, which is the mean of the abscissæ of the extremities, ought to be, very approximately, parallel to the chord; or, in other terms,

$$\frac{d}{dx}F\left(\frac{x_1+x_0}{2}\right)=\frac{F(x_1)-F(x_0)}{x_1-x_0}.$$

This gives the following values of $\frac{dy}{dx}$:

	x.	$\frac{dy}{dx}$.
1.	1.108	+0.0008113
2.	3.130	0.0005686
3.	4.231	0.0003770
4.	6.237	0.0003497
5.	8.600	0.0002788
6.	9.840	0.0003131
7.	12.428	0.0002948
8.	13.940	0.0002914
9.	16.265	+0.0002594

From the general course of these values of $\frac{dy}{dx}$, it may be gathered that this function, at first, diminishes rapidly, afterwards more slowly, and then tends, with higher values of x, to become nearly constant. But while this is the conclusion from the *tout ensemble*, a comparison of some of the values contradicts it. Thus, from 1, 2 and 3, while $\frac{dy}{dx}$ diminishes 0.0002427 in an interval 2.0 in x, it afterwards diminishes 0.0001916 in an interval 1.1 of x. All attempts then to represent these data by a curve, without singular points, must, evidently, show large errors.

For the discussion of the second class of experiments, let us assume that α has the signification we have given it in the general formula for V. Then the volume remaining the same, if P_0 and P_1 denote the tensions observed respectively at 0° and 100°, we have

$$\frac{P_1}{P_0} = \frac{1 + 100\,\alpha' + (\alpha + 100\,\alpha'')\,P_1^\beta}{1 + \alpha P_0^\beta},$$

$\frac{P_1}{P_0}$ is the quantity Regnault has designated by $1+100\alpha$, let us denote it by A; then if, for convenience, we put

$$\gamma = 1 + 100\alpha', \quad \gamma' = \alpha + 100\alpha'',$$

each determined value of A will give the equation of condition

$$\gamma + P_1^\beta . \gamma' = A + AP_0^\beta . \alpha.$$

The following are Regnault's determinations of A augmented, in general, by 0.00018, for the reason we adopt the mean coefficient 0.00018153 for the expansion of mercury between 0° and 100°, found by this experimenter, instead of the value $\frac{1}{5550}$ used by him (see Note, p. 31 of the volume); the last column contains the page of the volume, where the experiments may be found.

P_0. m	P_1. m	A.	No. Obs.	Obs.—Cal.	Page.
0.110	0.149	1.36500	10	—0.00012	99
0.174	0.237	1.36531	3	—0.00004	99
0.266	0.362	1.36560	2	—0.00003	99
0.375	0.510	1.36598	4	+0.00005	99
0.548	0.746	1.36673	3	+0.00038	57
0.756	0.7535	1.36724	4	+0.00035	66
0.557	0.754	1.36651	18	+0.00014	43
0.656	0.757	1.36641	14	—0.00022	33
0.747	1.016	1.36663	3	—0.00014	58
0.771	1.049	1.36696	11	+0.00014	51
1.678	2.286	1.36778	2	—0.00059	109
1.693	2.306	1.36818	4	—0.00021	109
2.526	2.517	1.36962	2	—0.00018	114
2.622	2.614	1.36982	2	—0.00011	114
2.144	2.924	1.36912	2	+0.00007	109
3.656	4.992	1.37109	4	+0.00031	109

Adopting, for convenience, as an unknown in the place of γ,

$$x = \gamma + \gamma' - 1.367,$$

we have the following equations, to each of which we attribute a weight equal to a tenth of the number of experiments it is founded upon:

$$\begin{aligned}
x - 0.7086\gamma' - 0.3268\alpha &= -0.00200\\
x - 0.6064\gamma' - 0.4398\alpha &= -0.00169\\
x - 0.4821\gamma' - 0.5790\alpha &= -0.00140\\
x - 0.3534\gamma' - 0.7237\alpha &= -0.00102\\
x - 0.1728\gamma' - 0.9258\alpha &= -0.00027\\
x - 0.1674\gamma' - 1.1410\alpha &= +0.00024\\
x - 0.1670\gamma' - 0.9354\alpha &= -0.00049\\
x - 0.1650\gamma' - 1.040\ \alpha &= -0.00059\\
x + 0.0105\gamma' - 1.131\ \alpha &= -0.00037\\
x + 0.0317\gamma' - 1.155\ \alpha &= -0.00004\\
x + 0.708\ \gamma' - 1.913\ \alpha &= +0.00078\\
x + 0.718\ \gamma' - 1.924\ \alpha &= +0.00118\\
x + 0.818\ \gamma' - 2.496\ \alpha &= +0.00262\\
x + 0.863\ \gamma' - 2.558\ \alpha &= +0.00282\\
x + 1.004\ \gamma' - 2.243\ \alpha &= +0.00212\\
x + 1.834\ \gamma' - 3.175\ \alpha &= +0.00409
\end{aligned}$$

The derived normal equations, for determining x and γ', are

$$\begin{aligned}
x - 0.0047\gamma' - 1.162\alpha &= -0.000144,\\
-0.0415x + 2.9547\gamma' - 3.373\alpha &= +0.007074,
\end{aligned}$$

whence

$$x = -0.000133 + 1.168\alpha, \quad \gamma' = +0.002392 + 1.158\alpha,$$

and

$$\alpha' = +0.00364475 + 0.00010\alpha, \quad \alpha'' = +0.00002392 + 0.00158\alpha.$$

The equation which determines α has already been obtained from the discussion of the first class of experiments; it is

$$\frac{\alpha + 4.747\alpha''}{1 + 4.747\alpha'} = -0.0024286.$$

The last three equations being solved, we gather that the volume of any mass of air is represented by the formula

$$V = \frac{K}{P}\left[1 + \alpha P^{\beta} + (\alpha' + \alpha'' P^{\beta})\,T\right],$$

in which

$$\begin{aligned}
\alpha &= -0.002565, & \alpha' &= +0.0036445,\\
\alpha'' &= +0.00001987, & \beta &= 0.64776.
\end{aligned}$$

How well the second class of experiments is satisfied by this formula may be seen from the numbers in the column headed Obs.–Cal.

If we have $T = \frac{0.002565}{0.00001987} = 129^\circ.1$, V takes the form

$$V = \frac{K}{P}.$$

Hence we have the noteworthy result that:

About the temperature 130°, *air follows quite exactly the law of Mariotte.*

For the following temperatures and pressures the volume vanishes:

T.	P.
	m
0°	9995.49
— 50	4420.13
—100	2048.00
—150	896.26
—200	314.23

These numbers may be regarded as indications of the magnitude of pressure necessary for the condensation of air. The table is in accordance with the well-known fact that reduction of temperature facilitates condensation.

A table is given below which will be found useful in the application of of the formula. It contains the functions $\log(1 + \alpha P^\beta)$ and $\frac{\alpha' + \alpha'' P^\beta}{1 + \alpha P^\beta}$, the latter being the coefficient of expansion under a constant pressure.

As an example, let us suppose that the volume of a mass of air has been observed under the pressure $2^m.5$ and the temperature 20°; it is required to find the factor necessary for reducing it to the pressure $0^m.76$ and temperature 0°. From the table we get 3.07064. By employing the ordinary formula with the coefficient 0.003665 of expansion, there is obtained 3.06482, which differs from the preceding by about a 525$^{\text{th}}$ part.

Rigorously, observations of pressure made in localities having an intensity of gravity different from that which prevails at Regnault's laboratory ought to be multiplied by the ratio of the former to the latter. The latitude of Regnault's laboratory is stated at 48° 50′ 14″, the altitude above sea level at 60^m, and the intensity of gravity at $9^m.8096$.

P.	$\log(1+\alpha P^{\beta})$.		Coeff. of Exp.	
0^m.0	0.000000	251	0.0036445	66
0 .1	9.999749	142	36511	37
0 .2	9.999607	118	36548	31
0 .3	9.999489	105	36579	28
0 .4	9.999384	96	36607	25
0 .5	9.999288	89	36632	24
0 .6	9.999199	84	36656	22
0 .7	9.999115	80	36678	20
0 .8	9.999035	77	36698	21
0 .9	9.998958	73	36719	19
1 .0	9.998885	71	36738	18
1 .1	9.998814	69	36756	19
1 .2	9.998745	67	36775	18
1 .3	9.998678	65	36793	16
1 .4	9.998613	64	36809	17
1 .5	9.998549	62	36826	16
1 .6	9.998487	61	36842	16
1 .7	9.998426	59	36858	16
1 .8	9.998367	59	36874	15
1 .9	9.998308	57	36889	15
2 .0	9.998251	272	36904	73
2 .5	9.997979	254	36977	67
3 .0	9.997725	240	37044	62
3 .5	9.997485	228	37106	60
4 .0	9.997257	218	37166	58
4 .5	9.997039	210	37224	56
5 .0	9.996829	203	37280	53
5 .5	9.996626	196	37333	52
6 .0	9.996430	191	37385	51
6 .5	9.996239	186	37436	49

P.	$\log(1+\alpha P^{\beta})$.		Coeff. of Exp.	
7^m.0	9.996053	181	0.0037485	48
7 .5	9.995872	177	37533	47
8 .0	9.995695	174	37580	46
8 .5	9.995521	169	37626	45
9 .0	9.995352	167	37671	44
9 .5	9.995185	164	37715	43
10 .0	9.995021	161	37758	43
10 .5	9.994860	158	37801	42
11 .0	9.994702	156	37843	42
11 .5	9.994546	153	37885	40
12 .0	9.994393	151	37925	40
12 .5	9.994242	149	37965	40
13 .0	9.994093	147	38005	39
13 .5	9.993946	146	38044	39
14 .0	9.993800	143	38083	38
14 .5	9.993657	142	38121	38
15 .0	9.993515	141	38159	37
15 .5	9.993374	138	38196	37
16 .0	9.993236	138	38233	36
16 .5	9.993098	136	38269	37
17 .0	9.992962	134	38306	36
17 .5	9.992828	133	38342	35
18 .0	9.992695	132	38377	35
18 .5	9.992563	131	38412	35
19 .0	9.992432	129	38447	35
19 .5	9.992303	129	38482	34
20 .0	9.992174	127	38516	34
20 .5	9.992047	126	38550	34
21 .0	9.991921	126	38584	34
21 .5	9.991795		38618	

MEMOIR NO. 31.

On Dr. Weiler's Secular Acceleration of the Moon's mean Motion.

(Astronomische Nachrichten, Vol. 91, pp. 251-254, 1878.)

Dr. Weiler's conclusions are, in general, not admissible because the expressions he gives for the forces X, Y and Z* are incorrect. It is well known that the attraction of a body, whatever may be its bounding surface and law of interior density, always admits a potential function W, such that

$$X = \frac{\partial W}{\partial x}, \quad Y = \frac{\partial W}{\partial y}, \quad Z = \frac{\partial W}{\partial z}.$$

But if we form the expression

$$Xdx + Ydy + Zdz$$

from Dr. Weiler's values of X, Y and Z, it is found to be not an exact differential: hence these values are erroneous.

They appear to have been derived by some illegitimate transformations from the formulas in the *Mécanique Céleste*, Tom. II, p. 22. After changing to Dr. Weiler's notation, Laplace's expressions for the attraction of a homogeneous ellipsoid of revolution become

$$X = -\frac{3hx}{k'^3}\int_0^1 \frac{u^2du}{\left(1+\frac{\lambda u^2}{k'^2}\right)^2}, \quad Y = \frac{3hy}{k'^3}\int_0^1 \frac{u^2du}{\left(1+\frac{\lambda u^2}{k'^2}\right)^2}, \quad Z = \frac{3hz}{k'^3}\int_0^1 \frac{u^2du}{1+\frac{\lambda u^2}{k'^2}},$$

where k' is given by the equation

$$2k'^2 = r^2 + \lambda - \sqrt{(r^2-\lambda)^2 + 4\lambda z^2}.$$

But Dr. Weiler seems to have put $k' = r$. This cannot be done for the k' which is outside of the sign of integration, without losing some part of the attraction which is of the order of the small quantity λ.

Hansen (*Fundamenta Nova*, pp. 1–16) has elaborated this matter with great generality and much elegance. From this source we learn that the proper expression for the potential function of the action between the earth and moon is

$$W = \frac{\varkappa(M+m)}{r}\left[1 + \tfrac{1}{2}\frac{A+B+C}{Mr^2} - \tfrac{3}{2}\frac{Ax^2+By^2+Cz^2}{Mr^4}\right],$$

* Astronomische Nachrichten, Vol. 90, pp. 372–373.

where A, B and C are the moments about the axes of x, y and z, supposed to coincide with the principal axes of rotation. In getting this expression, no assumption respecting the bounding surface or law of density of the earth is necessary; it is only assumed that terms of the third and higher orders with respect to the ratio of the dimensions of the earth to the radius-vector of the moon may be neglected.

Very nearly we have $B = A$, and, if this assumption is adopted, W takes the simpler form

$$W = \frac{\varkappa(M+m)}{r}\left[1 + \frac{1}{2}\frac{C-A}{Mr^2}\left(1 - 3\frac{z^2}{r^2}\right)\right].$$

If we put $k = \varkappa(M+m)$, and $\alpha = \frac{3}{2}\frac{C-A}{Ma_1^2}$, α will be a constant independent of the linear and time units; and measurements of arcs of the meridian, of the length of the second's pendulum, and the data afforded by the phenomena of precession and nutation, show that its value is very approximately $\alpha = 0.0016395$.

The expressions of the forces, which ought to be substituted for those given by Dr. Weiler, are then

$$X = -\frac{kx}{r^3}\left[1 + \alpha\frac{a_1^2}{r^2}\left(1 - 5\frac{z^2}{r^2}\right)\right],$$

$$Y = -\frac{ky}{r^3}\left[1 + \alpha\frac{a_1^2}{r^2}\left(1 - 5\frac{z^2}{r^2}\right)\right],$$

$$Z = -\frac{kz}{r^3}\left[1 + \alpha\frac{a_1^2}{r^2}\left(3 - 5\frac{z^2}{r^2}\right)\right].$$

MEMOIR No. 32.

Researches in the Lunar Theory.*

(American Journal of Mathematics, Vol. I, pp. 5-26, 129-147, 245-260, 1878.)

When we consider how we may best contribute to the advancement of this much-treated subject, we cannot fail to notice that the great majority of writers on it have had before them, as their ultimate aim, the construction of Tables; that is, they have viewed the problem from the stand-point of practical astronomy rather than of mathematics. It is on this account that we find such a restricted choice of variables to express the position of the moon, and of parameters, in terms of which to express the coefficients of the periodic terms. Again, their object compelling them to go over the whole field, they have neglected to notice many minor points of great interest to the mathematician, simply because the knowledge of them was unnecessary for the formation of Tables. But the developments having now been carried extremely far, without completely satisfying all desires, one is led to ask whether such modifications cannot be made in the processes of integration, and such coördinates and parameters adopted, that a much nearer approach may be had to the law of the series, and, at the same time, their convergence augmented.

Now, as to choice of coördinates, it is known that, in the elliptic theory, the rectangular coördinates of a planet, relative to the central body, the axes being parallel to the axes of the ellipse described, can be developed, in terms of the time, in the following series:

$$x = a \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J^{(i-1)}_{\frac{ie}{2}} \cos ig \,,$$

$$y = b \sum_{i=-\infty}^{i=+\infty} \frac{1}{i} J^{(i-1)}_{\frac{ie}{2}} \sin ig \,,$$

a and b being the semi-axes of the ellipse, e the eccentricity, g the mean anomaly and, for positive values of i, the Besselian function (in Hansen's notation)

$$J^{(i)}_{\lambda} = \frac{\lambda^i}{1.2\ldots i}\left[1 - \frac{\lambda^2}{1.(i+1)} + \frac{\lambda^4}{1.2.(i+1)(i+2)} - \ldots\right],$$

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* Communicated to the National Academy of Sciences at the session of April, 1877.

while, for negative values of i, we have

$$J_{\lambda}^{(-i)} = J_{-\lambda}^{(i)},$$

and, for the special case of $i = 0$, we put the indeterminate

$$\tfrac{1}{0} J_0^{(-1)} = -\tfrac{3}{2} e.$$

Here the law of the series is manifest, and the approximation can easily be carried as far as we wish. But the longitude and latitude, variables employed by nearly all the lunar-theorists, are far from having such simple expressions; in fact, their coefficients cannot be expressed finitely in terms of Besselian functions. And if this is true in the elliptic theory, how much more likely is a similar thing to be true when the complexity of the problem is increased by the consideration of disturbing forces? We are then justified in thinking that the coefficients of the periodic terms in the development of rectangular or quasi-rectangular coördinates are less complex functions of their parameters than those of polar coördinates. There is also another advantage in employing coördinates of the former kind; the differential equations are expressed in purely algebraic forms; while, with the latter, circular functions immediately present themselves. For these reasons I have not hesitated to substitute rectangular for polar coördinates.

Again, as to parameters, all those who have given literal developments, Laplace setting the example, have used the parameter m, the ratio of the sidereal month to the sidereal year. But a slight examination, even, of the results obtained, ought to convince any one that this is a most unfortunate selection in regard to convergence. Yet nothing seems to render this parameter at all desirable, indeed, the ratio of the synodic month to the sidereal year would appear to be more naturally suggested as a parameter. Some instances of slow convergence with the parameter m may be given from Delaunay's Lunar Theory; the development of the principal part of the coefficient of the evection in longitude begins with the term $\frac{15}{4} me$, and ends with the term $\frac{413277465931033}{15288238080} m^8 e$; again, in the principal part of the coëfficient of the inequality whose argument is the difference of the mean anomalies of the sun and moon, we find, at the beginning, the term $\frac{21}{4} mee'$, and, at the end, the term $\frac{1207454026843}{3538944} m^7 ee'$. It is probable that, by the adoption of some function of m as a parameter in place of this quantity, whose numerical value, in the case of our moon, should not greatly exceed

that of m, the foregoing large numerical coefficients might be very much diminished. And nothing compels us to use the same parameter throughout; one may be used in one class of inequalities, another in another, as may prove most advantageous. It is known what rapid convergence has been obtained in the series giving the values of logarithms, circular and elliptic functions, by simply adopting new parameters. Similar transformations, with like effects, are, perhaps, possible in the coëfficients of the lunar inequalities. However, as far as my experience goes, no useful results are obtained by experimenting with the present known developments; in every case it seems the proper parameter must be deduced from *a priori* considerations furnished in the course of the integration.

With regard to the form of the differential equations to be employed, although Delaunay's method is very elegant, and, perhaps, as short as any, when one wishes to go over the whole ground of the lunar theory, it is subject to some disadvantages when the attention is restricted to a certain class of lunar inequalities. Thus, do we wish to get only the inequalities whose coëfficients depend solely on m, we are yet compelled to develop the disturbing function R to all powers of e. Again, the method of integrating by undetermined coëfficients is most likely to give us the nearest approach to the law of the series; and, in this method, it is as easy to integrate a differential equation of the second order as one of the first, while the labor is increased by augmenting the number of variables and equations. But Delaunay's method doubles the number of variables in order that the differential equations may be all of the first order. Hence, in this disquisition, I have preferred to use the equations expressed in terms of the coördinates, rather than those in terms of the elements; and, in general, always to diminish the number of unknown quantities and equations by augmenting the order of the latter. In this way the labor of making a preliminary development of R in terms of the elliptic elements is avoided.

In the present memoir I propose, dividing the periodic developments of the lunar coördinates into classes of terms, after the manner of Euler in his last Lunar Theory,* to treat the following five classes of inequalities:

1. Those which depend only on the ratio of the mean motions of the sun and moon.
2. Those which are proportional to the lunar eccentricity.
3. Those which are proportional to the sine of the lunar inclination.
4. Those which are proportional to the solar eccentricity.
5. Those which are proportional to the solar parallax.

* *Theoria Motuum Lunæ, nova methodo pertractata. Petropoli,* 1772.

A general method will also be given by which these investigations may be extended so as to cover the whole ground of the lunar theory. My methods have the advantage, which is not possessed by Delaunay's that they adapt themselves equally to a special numerical computation of the coëfficients, or to a general literal development. The application of both procedures will be given in each of the just mentioned five classes of inequalities, so that it will be possible to compare them.

I regret that, on account of the difficulty of the subject and the length of the investigation it seems to require, I have been obliged to pass over the important questions of the limits between which the series are convergent, and of the determination of superior limits to the errors committed in stopping short at definite points. There cannot be a reasonable doubt that, in all cases, where we are compelled to employ infinite series in the solution of a problem, analysis is capable of being prefected to the point of showing us within what limits our solution is legitimate, and also of giving us a limit which its error cannot surpass. When the coördinates are developed in ascending powers of the time, or in ascending powers of a parameter attached as a multiplier to the disturbing forces, certain investigations of Cauchy afford us the means of replying to these questions. But when, for powers of the time, are substituted circular functions of it, and the coëfficients of these are expanded in powers and products of certain parameters produced from the combination of the masses with certain of the arbitrary constants introduced by integration, it does not appear that anything in the writings of Cauchy will help us to the conditions of convergence.

CHAPTER I.

Differential Equations.—Properties of motion derived from Jacobi's integral.

We set aside the action of the planets and the influence of the figures of the sun, earth and moon, together with the action of the last upon the sun, as also the product of the solar disturbing force on the moon by the small fraction obtained from dividing the mass of the earth by the mass of the sun. These are the same restrictions as those which Delaunay has imposed on his Lunar Theory contained in Vols. XXVIII and XXIX of the Memoirs of the Paris Academy of Sciences. The motion of the sun, about the earth, is then in accordance with the elliptic theory, and the ecliptic is a fixed plane.

Let us take a system of rectangular axes, having its origin at the centre of gravity of the earth, the axis of x being constantly directed toward the centre of the sun, the axis of y toward a point in the ecliptic 90° in advance

of the sun, and the axis of z perpendicular to the ecliptic. In addition, we adopt the following notation:

r' = the distance of the sun from the earth;
λ' = the sun's longitude;
μ = the sum of the masses of the earth and moon, measured by the velocity these masses produce by their action, in a unit of time, and at the unit of distance;
m' = the mass of the sun, measured in the same way;
n' = the mean angular velocity of the sun about the earth;
a' = the sun's mean distance from the earth.

In accordance with one of the above-mentioned restrictions we have the equation:

$$m' = n'^2 a'^3$$

The axes of x and y having a velocity of rotation in their plane, equal to $\frac{d\lambda'}{dt}$, it is evident that the square of the velocity of the moon, relative to the earth's centre, has for expression, in terms of the adopted coördinates,

$$\begin{aligned} 2T &= \left[\frac{dx}{dt} - y\frac{d\lambda'}{dt}\right]^2 + \left[\frac{dy}{dt} + x\frac{d\lambda'}{dt}\right]^2 + \frac{dz^2}{dt^2} \\ &= \frac{dx^2 + dy^2 + dz^2}{dt^2} + 2\frac{d\lambda'}{dt}\frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2}(x^2 + y^2). \end{aligned}$$

The potential function, in terms of the same coördinates, is

$$\Omega = \frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} + \frac{n'^2 a'^3}{\sqrt{[(r' - x)^2 + y^2 + z^2]}} - \frac{n'^2 a'^3}{r'^2}x.$$

If the second radical in this expression is expanded in a series proceeding according to descending powers of r' and the first term $\frac{n'^2 a'^3}{r'}$ omitted, since it disappears in all differentiations with respect to the moon's coördinates, the following expression is obtained:

$$\begin{aligned} \Omega = \frac{\mu}{\sqrt{(x^2 + y^2 + z^2)}} &+ n'^2 \frac{a'^3}{r'^3}\left[x^2 - \tfrac{1}{2}(y^2 + z^2)\right] \\ &+ \frac{n'^2}{a'}\frac{a'^4}{r'^4}\left[x^3 - \tfrac{3}{2}x(y^2 + z^2)\right] \\ &+ \frac{n'^2}{a'^2}\frac{a'^5}{r'^5}\left[x^4 - 3x^2(y^2 + z^2) + \tfrac{3}{8}(y^2 + z^2)^2\right] \\ &+ \frac{n'^2}{a'^3}\frac{a'^6}{r'^6}\left[x^5 - 5x^3(y^2 + z^2) + \tfrac{15}{8}x(y^2 + z^2)^2\right] \\ &+ \cdots\cdots \end{aligned}$$

Since the differential equations of motion are of the form

$$\frac{d}{dt}\cdot\frac{dT}{d\cdot\frac{d\varphi}{dt}}-\frac{dT}{d\varphi}=\frac{d\Omega}{d\varphi},$$

ϕ denoting, in succession, each of the variables which define the position of the moon, it is plain that the term

$$\tfrac{1}{2}\frac{d\lambda'^2}{dt^2}(x^2+y^2)$$

may be removed from T and added to Ω; and these modified quantities may be denoted by the symbols T' and Ω'. Then these equations may be written thus:

$$\begin{aligned}
\frac{d^2x}{dt^2}-2\frac{d\lambda'}{dt}\frac{dy}{dt}-\frac{d^2\lambda'}{dt^2}y&=\frac{d\Omega'}{dx},\\
\frac{d^2y}{dt^2}+2\frac{d\lambda'}{dt}\frac{dx}{dt}+\frac{d^2\lambda'}{dt^2}x&=\frac{d\Omega'}{dy},\\
\frac{d^2z}{dt^2}&=\frac{d\Omega'}{dz}.
\end{aligned}$$

When we wish to restrict our attention to the lunar inequalities which are independent of the solar parallax, all the terms, in the last expression of Ω, which are divided by r'^4, r'^5, r'^6, &c., may be omitted. In this case it will be seen that all the terms, introduced into the differential equations by the solar action, are linear in form, with variable, but known coëfficients, since $\frac{d\lambda'}{dt}$, $\frac{d^2\lambda'}{dt^2}$ and $\frac{a'^3}{r'^3}$ are known functions of t.

When all the inequalities, involving the solar eccentricity, are neglected, the equations admit an integral in finite terms. For, in this case, we have

$$\frac{d\lambda'}{dt}=n',\quad \frac{d^2\lambda'}{dt^2}=0,\quad r'=a',$$

and Ω' does not explicitly contain t; hence, multiplying the equations severally by the factors dx, dy, and dz, and adding the products, both members of the resulting equation are exact differentials. Integrating this equation, we have

$$\frac{dx^2+dy^2+dz^2}{2dt^2}=\Omega'+\text{a constant}.$$

This integral equation appears to have been first obtained by Jacobi.* As it will be frequently referred to in what follows, I shall take the liberty of calling it Jacobi's integral.

* *Comptes Rendus de l'Académie des Sciences de Paris.* Tom. III, p. 59.

If we take two imaginary variables

$$u = x + \sqrt{(y^2 + z^2)}\sqrt{-1},$$
$$s = x - \sqrt{(y^2 + z^2)}\sqrt{-1},$$

Ω has the following simple expression, being a function of two variables only:

$$\Omega = \frac{\mu}{\sqrt{us}} + \frac{n'^2 a'^3}{\sqrt{(r' - u)}\sqrt{(r' - s)}} - \frac{n'^2 a'^3}{2r'^3}(u + s).$$

If this is expanded in descending powers of r', and, as before, the term $\frac{n'^2 a'^3}{r'}$ omitted,

$$\Omega = \frac{\mu}{\sqrt{us}} + n'^2 \frac{a'^3}{r'^3}\left[\tfrac{3}{8}u^2 + \tfrac{1}{4}us + \tfrac{3}{8}s^2\right]$$
$$+ \frac{n'^2}{a'}\frac{a'^4}{r'^4}\left[\tfrac{5}{16}u^3 + \tfrac{3}{16}u^2 s + \tfrac{3}{16}us^2 + \tfrac{5}{16}s^3\right]$$
$$+ \frac{n'^2}{a'^2}\frac{a'^5}{r'^5}\left[\tfrac{35}{128}u^4 + \tfrac{5}{32}u^3 s + \tfrac{9}{64}u^2 s^2 + \tfrac{5}{32}us^3 + \tfrac{35}{128}s^4\right]$$
$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The additional variable, necessary to complete the definition of the moon's position, may be so taken that the expression of T may be simplified as much as possible. This expression may be written

$$2T = \frac{duds}{dt^2} - 4\frac{(ydz - zdy)^2}{(u - s)^2 dt^2} + 2\frac{d\lambda'}{dt}\frac{xdy - ydx}{dt} + \frac{d\lambda'^2}{dt^2}(us - z^2).$$

There does not seem to be any function of x, y and z, which, adopted as a new variable to accompany u and s, would reduce this to a very simple form. However, when we are engaged in determining the inequalities independent of the inclination of the lunar orbit, this transformation will be useful to us. For, in this case, $z = 0$, and the values of u and s become

$$u = x + y\sqrt{-1},$$
$$s = x - y\sqrt{-1},$$

and T is given by the equation

$$2T = \frac{duds}{dt^2} - \frac{d\lambda'}{dt}\frac{uds - sdu}{dt} + \frac{d\lambda'^2}{dt^2}us.$$

Although Ω is expressed most simply by the systems of coördinates we have just employed, the integration of the differential equations will be easier, if we suppose that the axes of x and y have a constant instead of a variable velocity of rotation, the axis of x being made to pass through the

mean position of the sun instead of the true. To obtain the expression for T correspondent to this supposition, it is necessary only to write n' for $\frac{d\lambda'}{dt}$ in the former values. As for Ω, it can be written thus

$$\Omega = \frac{\mu}{r} + \frac{n'^2 a'^3}{[r'^2 - 2r'S + r^2]^{\frac{1}{2}}} - \frac{n'^2 a'^3}{r'^2} S,$$

where

$r^2 = x^2 + y^2 + z^2 =$ the square of the moon's radius vector;
$S = x \cos \upsilon + y \sin \upsilon$;
$\upsilon =$ the solar equation of the centre.

This function being expanded in a series of descending powers of r', as before, we have

$$\begin{aligned}\Omega' = \frac{\mu}{r} &+ \tfrac{1}{2} n'^2 (x^2 + y^2) \\ &+ n'^2 \frac{a'^3}{r'^3} [\tfrac{3}{2} S^2 - \tfrac{1}{2} r^2] \\ &+ \frac{n'^2}{a'} \frac{a'^4}{r'^4} [\tfrac{5}{2} S^3 - \tfrac{3}{2} r^2 S] \\ &+ \frac{n'^2}{a'^2} \frac{a'^5}{r'^5} [\tfrac{35}{8} S^4 - \tfrac{15}{4} r^2 S^2 + \tfrac{3}{8} r^4] \\ &+ \frac{n'^2}{a'^3} \frac{a'^6}{r'^6} [\tfrac{63}{8} S^5 - \tfrac{35}{4} r^2 S^3 + \tfrac{15}{8} r^4 S] \\ &+ \ldots\ldots\ldots\end{aligned}$$

And the corresponding differential equations are

$$\begin{aligned}\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} &= \frac{d\Omega'}{dx}, \\ \frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} &= \frac{d\Omega'}{dy}, \\ \frac{d^2z}{dt^2} \qquad\qquad &= \frac{d\Omega'}{dz}.\end{aligned}$$

Thus much in reference to the equations under as general a form as we shall have occasion for in the present disquisition. We shall now suppose that they are reduced to as restricted a form as is possible without their becoming the equations of the elliptic theory; that is, we shall assume that the solar parallax and eccentricity and the lunar inclination vanish. With these simplifications, in the first system of coördinates,

$$T' = \frac{dx^2 + dy^2}{2dt^2} + n' \frac{xdy - ydx}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{(x^2 + y^2)}} + \tfrac{3}{2} n'^2 x^2;$$

and, in the second,

$$T' = \frac{duds}{2dt^2} - \frac{n'}{2}\frac{uds - sdu}{dt},$$

$$\Omega' = \frac{\mu}{\sqrt{us}} + \tfrac{3}{8}n'^2(u+s)^2.$$

And the differential equations, correspondent, are in the first case,

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2\right]x = 0,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0,$$

and, in the second,

$$\frac{d^2u}{dt^2} - 2n'\frac{du}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}u - \tfrac{3}{2}n'^2(u+s) = 0,$$

$$\frac{d^2s}{dt^2} + 2n'\frac{ds}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}s - \tfrac{3}{2}n'^2(u+s) = 0.$$

The Jacobian integral has severally the expressions

$$\frac{dx^2 + dy^2}{2dt^2} = \frac{\mu}{r} + \tfrac{3}{2}n'^2x^2 - C,$$

$$\frac{duds}{2dt^2} = \frac{\mu}{\sqrt{us}} + \tfrac{3}{8}n'^2(u+s)^2 - C.$$

The terms $-2n'\frac{dy}{dt}$, $2n'\frac{dx}{dt}$, &c., have been introduced into the equations by making the axes of coördinates movable; but since the putting of $n'=0$ makes the solar disturbing force vanish, there is no inconsistency in attributing them to the solar action. Then, in the case of the vanishing of this action, we have the equations of ordinary elliptic motion

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x = 0,$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = 0.$$

Thus, in the restricted case we consider, all the terms, added to the differential equations of motion by the solar action, are linear in form and have constant coëfficients. This, and the circumstance that t does not explicitly appear in the equations, are two advantages which are due to the particular selection of the variables x and y. If $\frac{\mu}{r^3}$ were constant, the equations would be linear with constant coëfficients and easily integrable.

The constants μ and n' can be made to disappear from the differential equations, if, instead of leaving the units of length and time arbitrary, we assume them so that $\mu = 1$, and $n' = 1$. The new unit of length, will then be equal to $\sqrt[3]{\frac{\mu}{n'^2}}$ units of the previous measurement. The equations, thus simplified, are

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} + \left[\frac{1}{r^3} - 3\right]x = 0,$$

$$\frac{d^2y}{dt^2} - 2\frac{dx}{dt} + \frac{1}{r^3}y = 0,$$

with their integral

$$\frac{dx^2 + dy^2}{dt^2} = \frac{2}{r} + 3x^2 - 2C.$$

It will be perceived that, in this way, we make the differential equations absolutely the same for all cases of the satellite problem.

Let us put $\rho = \frac{dx}{dt}$, then

$$\frac{dy}{dt} = \left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}},$$

$$\frac{d\rho}{dt} = 2\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}} - \left[\frac{1}{r^3} - 3\right]x.$$

Or, by making y the independent variable,

$$\frac{dx}{dy} = \frac{\rho}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}},$$

$$\frac{d\rho}{dy} = 2 - \frac{\left[\frac{1}{r^3} - 3\right]x}{\left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}}}.$$

The problem is then reduced to the integration of two differential equations of the first order. Were this accomplished, and ρ eliminated from the two integral equations, we should have the equation of the orbit. If we put

$$W = 2x + \left[\frac{2}{r} + 3x^2 - 2C - \rho^2\right]^{\frac{1}{2}},$$

the differential equations can be written in the canonical form,

$$\frac{dx}{dy} = -\frac{dW}{d\rho},$$

$$\frac{d\rho}{dy} = \frac{dW}{dx}.$$

It may be worth while to notice also the single partial differential equation, to the integration of which our problem can be reduced. Returning to the arbitrary linear and temporal units, and for convenience, reversing the sign of C, if a function of x and y can be found satisfying the partial differential equation

$$\left[\frac{dV}{dx}+n'y\right]^2+\left[\frac{dV}{dy}-n'x\right]^2=\frac{2\mu}{\sqrt{(x^2+y^2)}}+3n'^2x^2+2C,$$

and involving a single arbitrary constant h, distinct from that which can be joined to it by addition, the intermediate integrals of the problem will be

$$\frac{dx}{dt}=\frac{dV}{dx}+n'y, \qquad \frac{dy}{dt}=\frac{dV}{dy}-n'x,$$

and the final integrals

$$\frac{dV}{dh}=a, \qquad \frac{dV}{dC}=t+c,$$

a and c being two additional arbitrary constants. The truth of this will be evident if we differentiate the four integral equations with respect to t and compare severally the results with the partial differential coëfficients of the partial differential equation with respect to x, y, h and C.

Although, in this manner, the problem seems reduced to its briefest terms, yet, when we essay to solve it, setting out with this partial differential equation, we are led to more complex expressions than would be expected. It would be advisable, in this method of proceeding, to substitute polar for rectangular coördinates, or to put

$$x=r\cos\varphi, \qquad y=r\sin\varphi.$$

The partial differential equation, thus transformed, is

$$\frac{dV^2}{dr^2}+\left[\frac{1}{r}\frac{dV}{d\varphi}-n'r\right]^2=\frac{2\mu}{r}+\tfrac{3}{2}n'^2r^2+2C+\tfrac{3}{2}n'^2r^2\cos2\varphi.$$

This would have to be integrated by successive approximations, and it is found that this method, which at first sight, seems likely to afford a briefer solution of the problem, because but one unknown quantity was to be determined, and this free from the variable t, and involving only half of the number of arbitrary constants introduced by integration, when developed, leads to as complex operations as the older methods, and, moreover, has the disadvantage of giving results which need prolix transformations before the coördinates can be exhibited in terms of the time.

Although we shall make no use of the equations in terms of polar coördinates, they may be given here, for the sake of some special properties they possess in this form. They are

$$r\frac{d^2r}{dt^2} - r^2\frac{d\varphi^2}{dt^2} - 2n'r^2\frac{d\varphi}{dt} + \frac{\mu}{r} - 3n'^2r^2\cos^2\varphi = 0,$$

$$\frac{d}{dt}\left[r^2\left(\frac{d\varphi}{dt} + n'\right)\right] + \tfrac{3}{2}n'^2r^2\sin 2\varphi \qquad = 0,$$

with their integral

$$\frac{dr^2 + r^2d\varphi^2}{dt^2} = \frac{2\mu}{r} + 3n'^2r^2\cos^2\varphi - 2C.$$

By dividing the second of the differential equations by r^2, the variables are separated, and λ denoting the longitude of the moon, we have

$$r = \frac{K}{\sqrt{\frac{d\lambda}{dt}}}\varepsilon^{-\frac{3}{4}n'^2\int\frac{\sin 2(\lambda-\lambda')}{\frac{d\lambda}{dt}}dt},$$

K being a constant. Thus, after the longitude is determined in terms of t, the radius vector is obtained by a quadrature. But it can also be found, without the necessity of an integration, by dividing the integral by r^2 and then eliminating the term $\frac{1}{r^2}\frac{dr^2}{dt^2}$ by means of its value derived from the second differential equation; in this way we get

$$\frac{\mu}{r^3} - \frac{C}{r^2} = \tfrac{1}{8}\left[\frac{\frac{d^2\varphi}{dt^2} + \frac{3}{2}n'^2\sin 2\varphi}{\frac{d\varphi}{dt} + n'}\right]^2 + \tfrac{1}{2}\frac{d\varphi^2}{dt^2} - \tfrac{3}{2}n'^2\cos^2\varphi.$$

As we desire to make constant numerical application of the general theory, established in what follows, to the particular case of the moon, we delay here, for a moment, to obtain the numerical values of the three constants μ, n' and C. The value of μ may be derived either from the observed value of the constant of lunar parallax combined with the mean angular motion of the moon, or from the intensity of gravity at the earth's surface and the ratio of the moon's mass to that of the earth. We will adopt the latter procedure. The value of gravity at the equator, $g = 9.779741$ metres, the unit of time being the mean solar second. We propose, however, to take the mean solar day as the unit of time, and the equatorial radius of the earth as the linear unit. This number must then be multiplied by $\frac{86400^2}{6377397.15}$, (6377397.15 metres is Bessel's value of the equatorial radius.) Moreover,

the theory of the earth's figure shows that, in order to obtain the attractive force of the earth's mass, considered as concentrated at its centre of gravity, a second multiplication must be made by the factor 1.001818356. With our units then this force is represented by the number 11468.338: and the moon's mass being taken at $\frac{1}{81.52277}$ of the earth's, her attractive force is represented by the number 140.676. Consequently

$$\mu = 11609.014.$$

The sidereal mean motion of the sun in a Julian year is 1295977″.41516, whence

$$n' = 0.017202124.$$

The value of C might be obtained from the observed values of the moon's coördinates and their rates of variation at any time. However, as the eccentricity of the earth's orbit is not zero, C obtained in this manner would be found to undergo slight variations. The mean of all the values obtained in a long series of observations might be adopted as the proper value of this quantity when regarded as constant. But it is much easier to derive it approximately from the series

$$2C = (\mu n)^{\frac{2}{3}} \left[1 + 2\mathrm{m} - \frac{5}{6}\mathrm{m}^2 - \mathrm{m}^3 - \frac{1319}{288}\mathrm{m}^4 - \frac{67}{144}\mathrm{m}^5 - \frac{2879}{1296}\mathrm{m}^6 - \frac{1321}{1296}\mathrm{m}^7\right],$$

which will be established in the following chapter. Here n denotes the moon's sidereal mean motion, and m is put for $\frac{n'}{n - n'}$. In this formula the terms which involve the squares of the lunar eccentricity and inclination and of the solar parallax are neglected; this, however, is not of great moment, as, being multipled by at least m^2, they are of the fourth order with respect to smallness. The observations give $n = 0.22997085$, hence

$$C = 111.18883.$$

If it is proposed to assume the units of time and length so that μ and n' may both be unity, it will be found that the first is equal to 58.13236 mean solar days, and the second to 339.7898 equatorial radii of the earth. The corresponding value of C is 3.254440.

Let us now notice some of the properties of motion which can be derived from Jacobi's integral. This integral gives the square of the velocity relative to the moving axes of coördinates; and, as this square is necessarily positive, the putting it equal to zero gives the equation of the surface which

separates those portions of space, in which the velocity is real, from those in which it is imaginary. This equation is, in its most general form,

$$\frac{\mu}{\sqrt{(x^2+y^2+z^2)}} + \frac{n'^2a'^3}{\sqrt{[(a'-x)^2+y^2+z^2]}} = C + \tfrac{3}{2}n'^2a'^2 - \frac{n'^2}{2}[(a'-x)^2+y^2],$$

which is seen to be of the 16th degree. As y and z enter it only in even powers, the surface is symmetrically situated with respect to the planes of xy and xz. The left member is necessarily positive, (the folds of the surface, for which either or both the radicals receive negative values, are excluded from consideration), hence the surface is inclosed within the cylinder whose axis passes through the centre of the sun perpendicularly to the ecliptic, and whose trace on this plane is a circle of the radius

$$a'\sqrt{\left(3+\frac{2C}{n'^2a'^2}\right)}.$$

As, in general, the second term of the quantity, under the radical sign, is much smaller than the first, this radius is, quite approximately $\sqrt{3}a'$. Thus, in the case of our moon, assuming $\frac{1}{a'} = \sin 8''.848$, we have this radius $=$ $\sqrt{3.001383}a'$. It is evident that, for all points without this cylinder, the velocity is real; and as, for large values of z, whether positive or negative, the left member of the equation becomes very small, it is plain that the cylinder is asymptotic to the surface. Every right line, perpendicular to the ecliptic, intersects the surface not more than twice, at equal distances from this plane, once above and once below.

Let us now find the trace of the surface on the plane of xy. Putting ρ for the distance of a point on this trace from the centre of the sun,

$$\rho^2 = (a'-x)^2 + y^2,$$

and it is evident that the cubic equation,

$$\rho^3 = a'^2\left(3+\frac{2C}{n'^2a'^2}\right)\rho - 2a'^3,$$

will give the limits between which the values of ρ can oscillate. If C is negative, this equation has but one real root which is negative; consequently, in this case, the surface has no intersection with the plane of xy. But, in all the satellite systems we know, C is positive, and this condition is probably necessary to insure stability. Hence we shall restrict our attention to the case where C is positive. Then all the roots of the equation are real,

and two are positive. It is between the latter roots that ρ must always be found. To compute them, we derive the auxiliary angle θ from the formula

$$\sin\theta = \left[1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right]^{-\frac{3}{2}},$$

or, since θ differs but little from 90°, with more readiness from

$$\cos^2\theta = \frac{2\frac{C}{n'^2a'^2}\left[1 + \tfrac{2}{3}\frac{C}{n'^2a'^2} + \tfrac{4}{27}\frac{C^2}{n'^4a'^4}\right]}{\left[1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right]^3},$$

or, as $\frac{C}{n'^2a'^2}$ is a small quantity, with sufficient approximation from

$$\cos\theta = \frac{\sqrt{2\frac{C}{n'^2a'^2}}}{1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}}$$

The two roots are then

$$\rho_1 = 2a'\sqrt{\left(1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right)}\sin\frac{\theta}{3},$$

$$\rho_2 = 2a'\sqrt{\left(1 + \tfrac{2}{3}\frac{C}{n'^2a'^2}\right)}\sin\left(60° - \frac{\theta}{3}\right).$$

The trace of the surface on the plane of xy is then wholly comprised in the annular space between the two circles described from the centre of the sun as centre with the radii ρ_1 and ρ_2. Moreover, as in most satellite systems we have $\frac{\mu}{n'^2a'^3}$ equal to a very small fraction, $\left(\text{for our moon } \frac{\mu}{n'^2a'^3} = \frac{1}{322930.2}\right)$, it is plain that, for points whose distance from the earth is comparable with their distance from the sun, the trace is approximately coïncident with these circles. For the term $\frac{\mu}{r}$, in the equation, may then be neglected in comparison with the other terms.

In the case of our moon there is foumd

$$\theta = 87° \; 52' \; 11''.53,$$

and hence

$$\rho_1 = 22815.15, \qquad \rho_2 = 23816.09,$$

and, if r and ρ are regarded as the variables defining the position of a point in the plane xy, the following table gives some corresponding values of these

quantities, for each of the two branches of the trace approximating severally to the two circles.

r.	ρ.	r.	ρ.
433.3257	22878.69	439.7922	23751.81
450	22876.17	450	23753.37
500	22869.68	500	23760.04
600	22860.13	600	23769.85
1000	22841.59	1000	23788.87
10000	22817.70	10000	23813.43
46127.70	22815.68	47127.55	23815.53

The first and last values correspond to the four points where the curves intersect the axis of x on the hither and thither side of the sun. It will be seen that the approximation of the branches to the circles is quite close, except in the vicinity of the earth, where there is a slight protruding away from them.

In addition to these two branches of the trace, there is, in the case where C exceeds a certain limit, a third closed one about the origin much smaller than the former. As the coördinates of points in this branch are small fractions of a', its equation may be written, quite approximately,

$$\frac{\mu}{r} = C - \tfrac{3}{2} n'^2 x^2.$$

It intersects the axis of y at a distance from the origin very nearly

$$y_0 = \frac{\mu}{C},$$

and the axis of x at points whose coördinates are the smallest (without regard to sign) roots of the equations

$$\frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} = C + \tfrac{3}{2} n'^2 a'^2 - \tfrac{1}{2} n'^2 (a' - x)^2,$$

$$-\frac{\mu}{x} + \frac{n'^2 a'^3}{a' - x} = C + \tfrac{3}{2} n'^2 a'^2 - \tfrac{1}{2} n'^2 (a' - x)^2.$$

For the moon these quantities have the values

$$y_0 = 104.408, \qquad x_1 = -109.655, \qquad x_2 = +109.694.$$

This branch then does not differ much from a circle having its centre at the origin, more closely it approximates to the ellipse whose major axis $= x_2 - x_1$, and minor axis $= 2y_0$.

The value of the coördinate z, for the single intersection of the surface with the axis of z above the plane of xy, is given by the single positive root of the equation

$$\frac{\mu}{z} + \frac{n'^2 a'^3}{\sqrt{(a'^2 + z^2)}} = C + n'^2 a'^2.$$

For the moon the numerical value of this root is

$$z_0 = 102.956\,.$$

The intersection of the surface with the perpendicular to the plane of xy passing through the centre of the sun is, in like manner, given by the equation

$$\frac{\mu}{\sqrt{(a'^2 + z^2)}} + \frac{n'^2 a'^3}{z} = C + \tfrac{3}{2} n'^2 a'^2,$$

having but a single positive root, which is nearly

$$z_0 = \frac{\tfrac{2}{3} a'}{1 + \tfrac{2}{3} \dfrac{C}{n'^2 a'^2}},$$

or, with less exactitude,

$$z_0 = \tfrac{2}{3} a'.$$

From this investigation it is possible to get a tolerably clear idea of the form of this surface. When C exceeds a certain limit, it consists of three separate folds. The first being quite small, relatively to the other two, is close, surrounds the earth and somewhat resembles an ellipsoid whose axes have been given above. The second is also closed, but surrounds the sun, and has approximately the form of an ellipsoid of revolution, the semiaxis in the plane of the ecliptic being somewhat less than a', and the semiaxis of revolution perpendicular to the ecliptic and passing through the sun being about two-thirds of this. This fold has a protuberance in the portion neighboring the earth. The third fold is not closed, but is asymptotic to the cylinder mentioned at the beginning of the investigation of the surface. Like the second, it also is nearly of revolution about an axis passing through the centre of the sun and perpendicular to the ecliptic. The radius of its trace on the ecliptic is about as much greater than a', as the radius of the trace of the second fold falls short of that quantity. The fold has a protuberance in the portion neighboring the earth, and which projects towards this

body. The whole fold resembles a cylinder bent inwards in a zone neighboring the ecliptic.

What modifications take place in these folds when the constants involved in the equation of the surface are made to vary, will be clearly seen from the following exposition. Let us, for brevity, put

$$h = 3 + 2\frac{C}{n'^2a'^2},$$

and, for the moment, adopt a', the distance of the earth from the sun, as the linear unit, and transfer the origin to the centre of the sun, and moreover put

$$\gamma = \frac{\mu}{n'^2a'^3}.$$

Then the intersections of the surface, with the axis of x, will be given by the two roots of the equation

$$x^4 - x^3 - hx^2 + (h + 2 - 2\gamma)x - 2 = 0,$$

which lie between the limits 0 and 1; by the two roots of

$$x^4 - x^3 - hx^2 + (h + 2 + 2\gamma)x - 2 = 0,$$

which lie between 1 and $\sqrt{h}$; and by the two roots of

$$x^4 - x^3 - hx^2 + (h - 2 - 2\gamma)x + 2 = 0,$$

which lie between 0 and $-\sqrt{h}$.

Hence, if C diminishes so much that the first of these three equations has the two roots, lying between the mentioned limits, equal, the first fold will have a contact with the second fold; and if C fall below this limit, the roots become imaginary, and the two folds become one. Again, if C is diminished to the limit where the second equation has the mentioned pair of roots equal, the first fold will have a contact with the third; and when C is less than this, these two folds form but one. And when C is less than both these limits, there will be but one fold to the surface.

In the spaces inclosed by the first and second folds the velocity, relative to the moving axes of coördinates, is real; but, in the space lying between these folds and the third fold, it is imaginary; without the third fold it is again real. Thus, in those cases, where C and γ have such values that the three folds exist, if the body, whose motion is considered, is found at any time within the first fold, it must forever remain within it, and its radius

vector will have a superior limit. If it be found within the second fold, the same thing is true, but the radius vector will have an inferior as well as a superior limit. And if it be found without the third fold, it must forever remain without, and its radius vector will have an inferior limit.

Applying this theory to our satellite, we see that it is actually within the first fold, and consequently must always remain there, and its distance from the earth can never exceed 109.694 equatorial radii. Thus, the eccentricity of the earth's orbit being neglected, we have a rigorous demonstration of a superior limit to the radius vector of the moon.

Iu the cases, where C and γ have such values that the surface forms but one fold, Jacobi's integral does not afford any limits to the radius vector.

When we neglect the solar parallax and the lunar inclination, the preceding investigation is reduced to much simpler terms. The surface then degenerates into a plane curve, whose equation, of the sixth degree, is

$$\frac{\mu}{r} = C - \tfrac{3}{2} n'^2 x^2.$$

It is evidently symmetrical with respect to both axes of coördinates, and is contained between the two right lines, whose equations are

$$x = \pm \sqrt{\frac{2C}{3n'^2}},$$

and which are asymptotic to it. It intersects the axis of y, at two points, whose coördinates are

$$y = \pm \frac{\mu}{C}.$$

The cubic equation,

$$r^3 - \frac{2C}{3n'^2} r + \frac{2\mu}{3n'^2} = 0$$

gives the values of r, for which the curves intersect the axis of x. If

$$(2C)^{\frac{3}{2}} > 9\mu n',$$

this equation has two real roots between the limits 0 and $+\sqrt{\frac{2C}{3n'^2}}$. If

$$(2C)^{\frac{3}{2}} = 9\mu n',$$

these roots become equal. And if

$$(2C)^{\frac{3}{2}} < 9\mu n',$$

there are no real roots between these limits, and the curve has no intersection with the axis of x. The figures below exhibit the three varieties of this curve.

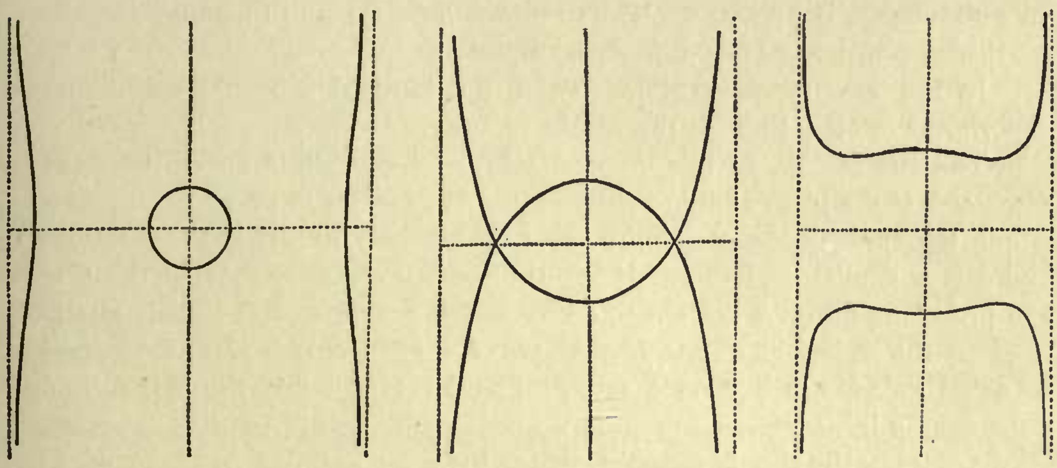


Fig. 1 represents the form of the curve in the case of our moon. In Fig. 2 we see that the small oval of Fig. 1 has enlarged and elongated itself so as to touch the two infinite branches; while, in Fig. 3, it has disappeared, the portions of the curve, lying on either side of the axis of x, having lifted themselves away from it, and the angles having become rounded off. In Fig. 1, the velocity is real within the oval, and also without the infinite branches, but it is imaginary in the portion of the plane lying between the oval and these branches. Hence, if the body be found, at any time, within the oval, it cannot escape thence, and its radius vector will have a superior limit; and, if it be found in one of the spaces on the concave side of the infinite branches, it cannot remove to the other, and its radius vector will have an inferior limit.

In the case represented in Fig. 2, the same things are true, but it seems as if the body might escape from the oval to the infinite spaces, or vice versa, at the points where the curve intersects the axis of x. However, at these points, the force, no less than the velocity, is reduced to zero. For the distance of these points from the origin is the positive root of the equation

$$3r^2 - \frac{2C}{3n'^2} = 0,$$

or

$$\frac{\sqrt{2C}}{3n'} = \frac{\sqrt[3]{9\mu n'}}{3n'},$$

and this value is the same as that given by the equation

$$\frac{\mu}{r^3} - 3n'^2 = 0\,.$$

In consequence the forces vanish at these two points, and thus we have two particular solutions of our differential equations.*

In the case represented in Fig. 3, the integral does not afford any superior or inferior limit to the radius vector.

The surface, or, in the more simple case, the plane curve, we have discussed, is the locus of zero velocity; and the surface or plane curve, upon which the velocity has a definite value, is precisely of the same character and has a similar equation. It is only necessary to suppose that the C of the preceding formulæ is augmented by half the square of the value attributed to the velocity. Thus, in the case of our moon, it is plain the curves of equal velocity will form a series of ovals surrounding the origin, and approaching it, and becoming more nearly circular as the velocity increases.

Applying the simple formulæ, where the solar parallax is neglected, to the moon, we find that the distance of the asymptotic lines, from the origin, is

$$\sqrt{\frac{2C}{3n'^2}} = 500.4992\,.$$

The distance of the points on the axis of x, at which the moon would remain stationary with respect to the sun, is

$$\sqrt[3]{\frac{\mu}{3n'^2}} = 235.5971\,.$$

If the auxiliary angle θ is derived from the equation

$$\sin\theta = \frac{9\mu n'}{(2C)^{3/2}}$$

we get

$$\theta = 32^\circ\ 49'\ 6''.63\,;$$

and the distances from the origin, at which the curve of zero velocity intersects the axis of x, are given by the two expressions

$$\frac{2\sqrt{2C}}{3n'}\sin\frac{\theta}{3}\,,$$

$$\frac{2\sqrt{2C}}{3n'}\sin\left(60^\circ - \frac{\theta}{3}\right),$$

and the numbers are 109.6772 and 435.5623. These values differ but little from the previous more general determinations.

* The corresponding solution, in the more general problem of three bodies, may be seen in the *Mécanique Céleste, Tom.* IV, p. 310.

CHAPTER II.

Determination of the inequalities which depend only on the ratio of the mean motions of the sun and moon.

If the path of a body, whose motion satisfies the differential equations

$$\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2\right] x = 0,$$
$$\frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} + \frac{\mu}{r^3} y \qquad = 0,$$

intersect the axis of x at right angles, the circumstances of motion, before and after the intersection, are identical, but in reverse order with respect to the time. That is, if t be counted from the epoch when the body is on the axis of x, we shall have

$$x = \text{function}\,(t^2), \qquad y = t\,.\,\text{function}\,(t^2).$$

For if, in the differential equations, the signs of y and t are reversed, but that of x left unchanged, the equations are the same as at first.

A similar thing is true if the path intersect the axis of y at right angles; for if the signs of x and t are reversed, while that of y is not altered, the equations undergo no change.

Now it is evident that the body may start from a given point on, and at right angles to, the axis of x, with different velocities; and that, within certain limits, it may reach the axis of y, and cross the same at correspondingly different angles. If the right angle lie between some of these, we judge, from the principle of continuity, that there is some intermediate velocity with which the body would arrive at and cross the axis of y at right angles.

The difficulty of this question does not permit its being treated by a literal analysis; but the tracing of the path of the body, in numerous special cases, by the application of mechanical quadratures to the differential equations, enables us to state the following circumstances:

If the body be projected at right angles to, and from a point on, the axis of x, whose distances from the origin is less than $0.33 \ldots \sqrt[3]{\frac{\mu}{n'^2}}$, there is at least one (near the limit there are two) value of the initial velocity, with which the body, in arriving at the axis of y, will cross it at right angles. Beyond this limit it appears no initial velocity will serve to make the body reach the axis of y under the stated condition.

If the body move from one axis to the other and cross both of them perpendicularly, it is plain, from the preceding developments, that its orbit

will be a closed curve symmetrical with respect to both axes. Thus is obtained a particular solution of the differential equations. While the general integrals involve four arbitrary constants, this solution, it is plain, has but two, which may be taken to be the distance from the origin at which the body crosses the the axis of x and the time of crossing.

Certain considerations, connected with the employment of Fourier's Theorem and the possibility of developing functions in infinite series of periodic terms, show that, in this solution, the coördinates of the body can be represented, in a convergent manner, by series of the following form:

$$x = A_0 \cos[\nu(t-t_0)] + A_1 \cos 3[\nu(t-t_0)] + A_2 \cos 5[\nu(t-t_0)] + \ldots,$$
$$y = B_0 \sin[\nu(t-t_0)] + B_1 \sin 3[\nu(t-t_0)] + B_2 \sin 5[\nu(t-t_0)] + \ldots,$$

where t_0 denotes the time the body crosses the axis of x, and $\frac{2\pi}{\nu}$ is the time of a complete revolution of the body about the origin. We may regard ν and t_0 as the arbitrary constants introduced by integration: the coefficients $A_0, A_1 \ldots . B_0, B_1 \ldots .$ are functions of μ, n' and ν.

For convenience sake we may put

$$A_i = a_i + a_{-i-1}, \qquad B_i = a_i - a_{-i-1}.$$

Then, τ being put for $\nu(t-t_0)$, the series, given above, may be written

$$x = \Sigma_i . a_i \cos(2i+1)\tau,$$
$$y = \Sigma_i . a_i \sin(2i+1)\tau,$$

the summation being extended to all integral values positive and negative zero included, for i. By adopting polar coördinates such that

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

and writing v for $\phi - \tau$, that is, for the excess of the true over the mean longitude of the moon, the last equations are equivalent to

$$r\cos v = \Sigma_i . a_i \cos 2i\tau,$$
$$r\sin v = \Sigma_i . a_i \sin 2i\tau.$$

In order to avoid the multiplication of series of sines and cosines, and reduce everything to an algebraic form, for x and y, we substitute the imaginary variables u and s, and put $\zeta = \varepsilon^{\tau\sqrt{-1}}$. We have then

$$u = \Sigma_i . a_i \zeta^{2i+1}, \qquad s = \Sigma_i . a_{-i-1} \zeta^{2i+1}.$$

ζ will always be employed as the independent variable in place of t or τ. Denoting the operation $\zeta \dfrac{d}{d\zeta} = -\sqrt{-1} \dfrac{d}{d\tau}$ by the symbol D, so that, in general,

$$D(a\zeta^i) = ia\zeta^i,$$

and taking the liberty of separating this symbol as if it were a multiplier, and moreover putting

$$\mathrm{m} = \frac{n'}{\nu} = \frac{n'}{n - n'}, \qquad \varkappa = \frac{\mu}{\nu^2},$$

the differential equations, determining u and s, given in the preceding chapter, may be written

$$\left[D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] u + \tfrac{3}{2}\mathrm{m}^2 s = 0,$$
$$\left[D^2 - 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] s + \tfrac{3}{2}\mathrm{m}^2 u = 0.$$

It will be noticed that either of these equations can be derived from the other by interchanging u and s and reversing the sign of m or D. We may also remind the reader that they determine rigorously all the parts of the lunar coördinates which depend only on the ratio of the mean motions of the sun and moon and on the lunar eccentricity. The Jacobian integral, in the present notation, is

$$Du\,.\,Ds + \frac{2\varkappa}{(us)^{\frac{1}{2}}} + \tfrac{3}{4}\mathrm{m}^2 (u + s)^2 = C.$$

The most ready method of getting the values of the coefficients a_i, is that of undetermined coefficients; the values of u and s, expressed by the preceding summations with reference to i, being substituted in the differential equations, the resulting coefficient of each power of ζ, in the left members, is equated to zero, which furnishes a series of equations of condition sufficient to determine all the quantities a_i. For this purpose we may evidently employ any two independent combinations of the three equations last written, and it will be advisable to form these combinations in such a manner that the process of deriving the equations of condition may be facilitated in the largest degree. Now it will be recognized that the presence of the term $\dfrac{\varkappa}{(us)^{\frac{3}{2}}}$, in one of the factors of the differential equations, is a hindrance to their ready integration, being the single thing which prevents them from being linear with constant coefficients. Hence we avail ourselves of the possibility of eliminating it. Multiplying the first differential equation by

s, and the second by u, and taking in succession, the sum and difference,

$$uD^2s + sD^2u - 2\mathrm{m}\,(uDs - sDu) - \frac{2\varkappa}{(us)^{1/2}} + \tfrac{3}{2}\mathrm{m}^2(u+s)^2 = 0\,,$$
$$uD^2s - sD^2u - 2\mathrm{m}\,(uDs + sDu) \qquad\qquad + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0\,,$$

then, adding to the first of these the integral equation, and retaining the second as it is, we have, as the final differential equations to be employed,

$$D^2(us) - Du\,.\,Ds - 2\mathrm{m}\,(uDs - sDu) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C\,,$$
$$D\,(uDs - sDu - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) \qquad = 0\,.$$

It must be pointed out, however, that these equations are not, in all respects, a complete substitute for the original equations. It will be seen that μ or $\varkappa$, an essential element in the problem, has disappeared from them, and that, in integration, an arbitrary constant, in excess of those admissible, will present itself. This will be eliminated by substituting the integrals found in one of the original differential equations, in which μ or $\varkappa$ is present; the result being an equation of condition by which the superfluous constant can be expressed in terms of μ and the remaining constants.

We remark that the left members of our differential equations are homogeneous and of two dimensions with respect to u and s. If the first were differentiated, the constant C would disappear, and both equations would be homogeneous in all their terms. This property renders them exceedingly useful when equations of condition are to be obtained between the coefficients of the different periodic terms of the lunar coördinates, and it is for this purpose that we have given them their present form.

From the signification of the symbol D,

$$Du = \Sigma_i\,.\,(2i+1)\,\mathrm{a}_i\zeta^{2i+1}, \qquad Ds = \Sigma_i\,.\,(2i+1)\,\mathrm{a}_{-i-1}\zeta^{2i+1},$$
$$D^2u = \Sigma_i\,.\,(2i+1)^2\,\mathrm{a}_i\zeta^{2i+1}, \qquad D^2s = \Sigma_i\,.\,(2i+1)^2\,\mathrm{a}_{-i-1}\zeta^{2i+1};$$

also

$$us = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$
$$u^2 = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{-i+j-1}]\,\zeta^{2j},$$
$$s^2 = \Sigma_j\,.\,[\Sigma_i\,.\,\mathrm{a}_i\mathrm{a}_{-i-j-1}]\,\zeta^{2j},$$
$$Du\,.\,Ds = -\,\Sigma_j\,.\,[\Sigma_i\,.\,(2i+1)(2i-2j+1)\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$
$$uDs - sDu = -\,2\Sigma_j\,.\,[\Sigma_i\,.\,(2i-j+1)\,\mathrm{a}_i\mathrm{a}_{i-j}]\,\zeta^{2j},$$

where the summations with reference to j have the same extension as those with reference to i. On substituting these expressions in the differential equations, and equating the general coefficients of ζ^{2j} to zero, we get

$$\Sigma_i\,.\,[(2i+1)(2i-2j+1) + 4j^2 + 4\,(2i-j+1)\,\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2]\,\mathrm{a}_i\mathrm{a}_{i-j}$$
$$+ \tfrac{9}{4}\mathrm{m}^2\Sigma_i\,.\,[\mathrm{a}_i\mathrm{a}_{-i+j-1} + \mathrm{a}_i\mathrm{a}_{-i-j-1}] = 0\,,$$
$$4j\Sigma_i\,.\,[2i - j + 1 + \mathrm{m}]\,\mathrm{a}_i\mathrm{a}_{i-j} - \tfrac{3}{2}\mathrm{m}^2\Sigma_i\,.\,[\mathrm{a}_i\mathrm{a}_{-i+j-1} - \mathrm{a}_i\mathrm{a}_{-i-j-1}] = 0,$$

which hold for all integral values of j both positive and negative except that, when $j=0$, the right member of the first equation is C instead of 0; but as the second equation is an identity for $j=0$, for the present this value of j will be excluded from consideration.

By multiplying the first equation by 2, and the second by 3, and taking in succession the difference and sum, the simpler forms are obtained,

$$\Sigma_i \,.\, [8i^2 - 8(4j-1)i + 20j^2 - 16j + 2 + 4(4i - 5j + 2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_i \mathrm{a}_{i-j} + 9\mathrm{m}^2 \Sigma_i \,.\, \mathrm{a}_i \mathrm{a}_{-i+j-1} = 0,$$

$$\Sigma_i \,.\, [8i^2 + 8(2j+1)i - 4j^2 + 8j + 2 + 4(4i + j + 2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_i \mathrm{a}_{i-j} + 9\mathrm{m}^2 \Sigma_i \,.\, \mathrm{a}_i \mathrm{a}_{-i-j-1} = 0.$$

These two equations are not distinct from each other, when negative, as well as positive values, are attributed to j. For if, in the expression under the first sign of summation in the first equation, we substitute, which is allowable, for i, $i-j$, and $-j$ for j throughout the equation, the result is identical with the second equation. This is explained by the fact that we get all the independent equations of condition, these equations are capable of furnishing, by attributing only positive values to j. Hence, allowing j to receive positive and negative values, all the equations of condition can be represented by a unique formula.

Although the number of these equations is infinite, and also that of the coefficients a_i, it is not difficult to see that the first ought to be regarded as one less than the second; and that, in consequence of the bi-dimensional character of the equations, they suffice to determine the ratio of any two of the quantities a_i in terms of m. It will be seen, from developments to be given shortly, that if m is regarded as a small quantity of the first order, a_i is of the $\pm\, 2i^{th}$ order. It will be advisable then to select a_0 as the coefficient to which to refer all the rest; and we shall have, in general,

$$\mathrm{a}_i = \mathrm{a}_0 F(\mathrm{m}).$$

The equations of condition, as written above, determine the a_i in pairs; that is, if we put $j=1$, we have the equations suitable for determining a_1 and a_{-1}, and, in general, the equations, as written, determine a_j and a_{-j}. And, as they involve both these quantities, it will be advantageous to eliminate approximately each in succession, as far as that can be done without depriving the equations of their bi-dimensional character.

By putting, in succession, in the terms under the first sign of summation, $i=0$ and $i=j$, it will be found that these equations contain, severally, the terms

$$[20j^2 - 16j + 2 - 4(5j-2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_0\mathrm{a}_{-j} + [-4j^2 - 8j + 2 - 4(j-2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_0\mathrm{a}_j,$$

$$[-4j^2 + 8j + 2 + 4(j+2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_0\mathrm{a}_{-j} + [20j^2 + 16j + 2 + 4(5j+2)\mathrm{m} + 9\mathrm{m}^2]\,\mathrm{a}_0\mathrm{a}_j,$$

which are the terms of principal moment in determining a_{-j} and a_j. Let us then multiply the first equation by

$$-4j^2+8j+2+4(j+2)\mathrm{m}+9\mathrm{m}^2,$$

and the second by

$$-20j^2+16j-2+4(5j-2)\mathrm{m}-9\mathrm{m}^2,$$

and, adding the products, divide the whole by

$$48j^2[2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2].$$

Then, adopting the notation

$$[j,i]=-\frac{i}{j}\frac{4(j-1)i+4j^2+4j-2-4(i-j+1)\mathrm{m}+\mathrm{m}^2}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

$$[j]\quad=-\frac{3\mathrm{m}^2}{16j^2}\frac{4j^2-8j-2-4(j+2)\mathrm{m}-9\mathrm{m}^2}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

$$(j)\quad=-\frac{3\mathrm{m}^2}{16j^2}\frac{20j^2-16j+2-4(5j-2)\mathrm{m}+9\mathrm{m}^2}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

the system of equations, which determines the coefficients a_i, is represented by the unique formula

$$\Sigma_i\cdot[[j,i]\,a_ia_{i-j}+[j]\,a_ia_{-i+j-1}+(j)\,a_ia_{-i-j-1}]=0,$$

where j must receive negative as well as positive values. It will be perceived that

$$[j,0]=0,\qquad [j,j]=-1;$$

hence the last equation is in a form suitable for determining the value of a_j. The quantities $[j,i]$, $[j]$ and (j) admit of being expressed in a simpler manner; thus

$$[j,i]=-\frac{i}{j}+\frac{4i(j-i)}{j}\frac{j-1-\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

whence

$$[j,i]+[-j,-i]=-\frac{2i}{j}+\frac{8i(j-i)}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

$$[j,i]-[-j,-i]=\quad\frac{8i(i-j)}{j}\frac{1+\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2},$$

in addition

$$[j]\quad=\quad\frac{27}{16j^2}\mathrm{m}^2-\frac{3}{4j^2}\frac{19j^2-2j-5-(j+11)\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2}\mathrm{m}^3,$$

$$(j)\quad=-\frac{27}{16j^2}\mathrm{m}^2+\frac{3}{4j^2}\frac{13j^2+4j-5+(5j-11)\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2}\mathrm{m}^3,$$

$$[j]+(-j)=-\frac{3}{2j}\frac{3j+1+2\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2}\mathrm{m}^3,$$

$$[j]-(-j)=\quad\frac{27}{8j^2}\mathrm{m}^2-\frac{3}{2j^2}\frac{16j^2-3j-5-(3j+11)\mathrm{m}}{2(4j^2-1)-4\mathrm{m}+\mathrm{m}^2}\mathrm{m}$$

In making a first approximation to the values of the coefficients, one of the terms of the equation may be omitted; for, when j is positive, the term $\Sigma_i \,.\, (j)\, a_i a_{-i-j-1}$ is a quantity four orders higher than that of the terms of the lowest order contained in the equation; and, when j is negative, the same thing is true of $\Sigma_i \,.\, [j]\, a_i a_{-i+j-1}$. Hence, with this limitation, the equation may be written in the two forms

$$\Sigma_i \,.\, [[j, i]\, a_i a_{i-j} + [j]\, a_i a_{-i+j-1}] \quad = 0,$$
$$\Sigma_i \,.\, [[-j, i]\, a_i a_{i+j} + (-j)\, a_i a_{-i+j-1}] = 0.$$

where j takes only positive values.

From these two equations, by omitting all terms but those of the lowest order, we derive the following series of equations, determining the coefficients to the first degree of approximation:

$$a_0 a_1 \;= [1]\, a_0 a_0,$$
$$a_0 a_{-1} = (-1)\, a_0 a_0,$$
$$a_0 a_2 \;= [2][a_0 a_1 + a_1 a_0] + [2, 1]\, a_1 a_{-1},$$
$$a_0 a_{-2} = (-2)\,[a_0 a_1 + a_1 a_0] + [-2, -1]\, a_1 a_{-1},$$
$$a_0 a_3 \;= [3][a_0 a_2 + a_1 a_1 + a_2 a_0] + [3, 1]\, a_1 a_{-2} + [3, 2]\, a_2 a_{-1},$$
$$a_0 a_{-3} = (-3)\,[a_0 a_2 + a_1 a_1 + a_2 a_0] + [-3, -1]\, a_{-1} a_2 + [-3, -2]\, a_{-2} a_1,$$
$$a_0 a_4 \;= [4][a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0] + [4, 1]\, a_1 a_{-3} + [4, 2]\, a_2 a_{-2} + [4, 3]\, a_3 a_{-1},$$
$$a_0 a_{-4} = (-4)\,[a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0] + [-4, -1]\, a_{-1} a_3 + [-4, -2]\, a_{-2} a_2$$
$$+ [-4, -3]\, a_{-3} a_1,$$

. . . .

The law of these equations is quite apparent, and they can easily be extended as far as desired. The first two give the values of a_1 and a_{-1}, the following two the values of a_2 and a_{-2} by means of the values of a_1 and a_{-1} already obtained, and so on, every two equations of the series giving the values of two coefficients by means of the values of all those which precede in the order of enumeration. A glance at the composition of these equations must convince us that all attempts to write explicity, even this approximate value of a_i, would be unsuccessful on account of the excessive multiplicity of the terms. However, they may be regarded, in some sense, as giving the law of this approximate solution, since they exhibit clearly the mode in which each coefficient depends on all those which precede it. As to the degree of approximation afforded by these equations, when the values are expanded in series of ascending powers of m, the first four terms are obtained correctly in the case of each coefficient. Thus a_1 and a_{-1} are affected with errors of the 6th order, a_2 and a_{-2} with errors of the 8th order, a_3 and a_{-3} with errors of the 10th order, and so on.

The values of these quantities can be determined either in the literal form, where the parameter m is left indeterminate, as has been done by Plana and Delaunay, or as numbers, which mode has been followed by all the earlier lunar-theorists and Hansen. In the latter case, one will begin by computing the numerical values of the quantities $[j, i]$, $[j]$ and (j), corresponding to the assumed value of m, for all necessary values of the integers i and j.

The great advantage of our equations consists in this, that we are able to extend the approximation as far as we wish, simply by writing explicitly the terms, our symbols giving the law of the coefficients. How rapid is the approximation in the terms of these equations will be apparent, when we say, that, after a certain number of terms are written, in order to carry this four orders higher, it is necessary to add to each of them only four new terms; and thereafter, every addition of four terms enables us to carry the approximation four orders farther.

The process which may be followed to obtain the values of the a_i with any desired degree of accuracy, is this:—the first approximate values will be got from the preceding group of equations until the a_i become of orders intended to be neglected; then one will recommence at the beginning, using the equations each augmented by the terms necessary to carry the approximation four orders higher; substituting in the new terms the values obtained from the first approximation, and, in the old, ascertaining what changes are produced by employing the more exact values instead of the first approximations. A second return to the beginning of the work will in like manner, push the degree of exactitude four orders higher. In this way any required degree of approximation may be attained.

Whatever advantage the present process may have over those previously employed is plainly due to the use of the indeterminate integers i and j, which, although much used in the planetary theories, no one seems to have thought of introducing into the lunar theory. This enables us to perform a large mass of operations once for all.

For the purpose of making evident the preceding assertions, and because we shall have occasion to use them, we write below the equations determining the coefficients a_i correct to quantities of the 13th order inclusive.

$$\begin{aligned}
a_0a_1 &= [1][a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] + (1)[a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] \\
&\quad + [1, -2]a_{-2}a_{-3} + [1, -1]a_{-1}a_{-2} + [1, 2]a_2a_1 + [1, 3]a_3a_2, \\
a_0a_{-1} &= [-1][a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] + (-1)[a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] \\
&\quad + [-1, -3]a_{-3}a_{-2} + [-1, -2]a_{-2}a_{-1} + [-1, 1]a_1a_2 + [-1, 2]a_2a_3, \\
a_0a_2 &= [2][2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] + (2)[2a_{-1}a_{-2} + 2a_0a_{-3} + 2a_1a_{-4}] \\
&\quad + [2, -2]a_{-2}a_{-4} + [2, -1]a_{-1}a_{-3} + [2, 1]a_1a_{-1} + [2, 3]a_3a_1 + [2, 4]a_4a_2,
\end{aligned}$$

$$
\begin{aligned}
a_0a_{-2} &= [-2][2a_{-1}a_{-2} + 2a_2a_{-5} + 2a_1a_{-4}] + (-2)\,[2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] \\
&\quad + [-2, -4]\,a_{-4}a_{-2} + [-2, -3]\,a_{-3}a_{-1} + [-2, -1]\,a_{-1}a_1 + [-2, 1]\,a_1a_3 . \\
&\quad + [-2, 2]\,a_2a_4 , \\
a_0a_3 &= [3][a_1^2 + 2a_0a_2 + 2a_{-1}a_3] + (3)\,[a_{-2}^2 + 2a_{-1}a_{-3} + 2a_0a_{-4}] \\
&\quad + [3, -1]\,a_{-1}a_{-4} + [3, 1]\,a_1a_{-2} + [3, 2]\,a_2a_{-1} + [3, 4]\,a_4a_1 , \\
a_0a_{-3} &= [-3][a_{-2}^2 + 2a_{-1}a_{-3} + 2a_0a_{-4}] + (-3)\,[a_1^2 + 2a_0a_2 + 2a_{-1}a_3] \\
&\quad + [-3, -4]\,a_{-4}a_{-1} + [-3, -2]\,a_{-2}a_1 + [-3, -1]\,a_{-1}a_2 + [-3, 1]\,a_1a_4 , \\
a_0a_4 &= [4][2a_1a_2 + 2a_0a_3 + 2a_{-1}a_4] + (4)\,[2a_{-2}a_{-3} + 2a_{-1}a_{-4} + 2a_0a_{-5}] \\
&\quad + [4, -1]\,a_{-1}a_{-5} + [4, 1]\,a_1a_{-3} + [4, 2]\,a_2a_{-2} + [4, 3]\,a_3a_{-1} + [4, 5]\,a_5a_1 , \\
a_0a_{-4} &= [-4][2a_{-2}a_{-3} + 2a_{-1}a_{-4} + 2a_0a_{-5}] + (-4)\,[2a_1a_2 + 2a_0a_3 + 2a_{-1}a_4] \\
&\quad + [-4, -5]\,a_{-5}a_{-1} + [-4, -3]\,a_{-3}a_1 + [-4, -2]\,a_{-2}a_2 + [-4, -1]\,a_{-1}a_3 \\
&\quad + [-4, 1]\,a_1a_5 , \\
a_0a_5 &= [5][a_2^2 + 2a_1a_3 + 2a_0a_4] + [5, 1]\,a_1a_{-4} + [5, 2]\,a_2a_{-3} + [5, 3]\,a_3a_{-2} + [5, 4]\,a_4a_{-1} , \\
a_0a_{-5} &= (-5)\,[a_2^2 + 2a_1a_3 + 2a_0a_4] + [-5, -4]\,a_{-4}a_1 \\
&\quad + [-5, -3]\,a_{-3}a_2 + [-5, -2]\,a_{-2}a_3 + [-5, -1]\,a_{-1}a_4 , \\
a_0a_6 &= [6][2a_2a_3 + 2a_1a_4 + 2a_0a_5] + [6, 1]\,a_1a_{-5} \\
&\quad + [6, 2]\,a_2a_{-4} + [6, 3]\,a_3a_{-3} + [6, 4]\,a_4a_{-2} + [6, 5]\,a_5a_{-1} , \\
a_0a_{-6} &= (-6)\,[2a_2a_3 + 2a_1a_4 + 2a_0a_5] \\
&\quad + [-6, -5]\,a_{-5}a_1 + [-6, -4]\,a_{-4}a_2 + [-6, -3]\,a_{-3}a_3 + [-6, -2]\,a_{-2}a_4 \\
&\quad + [-6, -1]\,a_{-1}a_5 .
\end{aligned}
$$

In the first approximation

$$
\begin{aligned}
a_1 &= [1]\,a_0 , \\
a_{-1} &= (-1)\,a_0 , \\
a_2 &= [1][2\,(2) + [2, 1]\,(-1)]\,a_0 , \\
a_{-2} &= [1][2\,(-2) + [-2, -1]\,(-1)]\,a_0 ,
\end{aligned}
$$

or, explicitly in terms of m,

$$
\begin{aligned}
a_1 &= \tfrac{3}{16}\,\frac{6 + 12m + 9m^2}{6 - 4m + m^2}\,m^2a_0 , \\
a_{-1} &= -\tfrac{3}{16}\,\frac{38 + 28m + 9m^2}{6 - 4m + m^2}\,m^2a_0 ,
\end{aligned}
$$

and, after some reductions,

$$
\begin{aligned}
a_2 &= \tfrac{27}{256}\,\frac{2 + 4m + 3m^2}{[6 - 4m + m^2][30 - 4m + m^2]}\left[238 + 40m + 9m^2 - 32\,\frac{29 - 35m}{6 - 4m + m^2}\right] m^4a_0 , \\
a_{-2} &= \tfrac{27}{64}\,\frac{2 + 4m + 3m^2}{[6 - 4m + m^2][30 - 4m + m^2]}\left[-28 - 7m + 24\,\frac{7 - m}{6 - 4m + m^2}\right] m^4a_0 .
\end{aligned}
$$

It is evident that, however far the approximation may be carried, the only quantities, involved as divisors in the values of the a_i, are the trinomials, whose general expression is

$$2\,(4j^2 - 1) - 4m + m^2 ,$$

or, particularizing, the series of divisors is

$$6-4m+m^2,$$
$$30-4m+m^2,$$
$$70-4m+m^2,$$
$$\cdots\cdots$$

It will be remarked that they differ only in their first terms, which are independent of m. Hence any expression, involving several divisors, can always be separated into several parts, each involving only one divisor, without any actual division by a trinomial in m. For instance,

$$\frac{1}{[6-4m+m^2][30-4m+m^2]}=\tfrac{1}{24}\frac{1}{6-4m+m^2}-\tfrac{1}{24}\frac{1}{30-4m+m^2},$$

$$\frac{1}{[6-4m+m^2]^2[30-4m+m^2]}=\tfrac{1}{24}\frac{1}{[6-4m+m^2]^2}$$
$$-\tfrac{1}{24^2}\frac{1}{6-4m+m^2}+\tfrac{1}{24^2}\frac{1}{30-4m+m^2}.$$

Moreover when, after this transformation, any numerator contains more or other powers of m than two consecutive powers, it is clear it may be reduced so as to contain only these by eliminating the higher powers through subtracting certain multiples of the divisor which appears in the denominator, or, in other words, the fraction may be treated as if it were improper.

From this we gather that the value of a_i can be expressed thus

$$\begin{aligned}\frac{a_i}{a_0}=M_0&+\frac{M_1}{6-4m+m^2}+\frac{M_2}{[6-4m+m^2]^2}+\frac{M_3}{[6-4m+m^2]^3}+\cdots\\&+\frac{N_1}{30-4m+m^2}+\frac{N_2}{[30-4m+m^2]^2}+\frac{N_3}{[30-4m+m^2]^3}+\cdots\\&+\frac{P_1}{70-4m+m^2}+\frac{P_2}{[70-4m+m^2]^2}+\frac{P_3}{[70-4m+m^2]^3}+\cdots\\&+\cdots\cdots\cdots\cdots\end{aligned}$$

where $M_0, M_1 \ldots N_1, N_2 \ldots P_1, P_2 \ldots$ are entire functions of m each of the form

$$Am^k+Bm^{k+1}.$$

The advantage of this method of treatment consists in that nothing, which is given by the successive approximations, would be lost, as must be the case when the values are expanded in series of ascending powers of m. The preceding expressions, when put into this form, become

$$\frac{a_1+a_{-1}}{a_0}=-3\frac{2+m}{6-4m+m^2}m^2,$$

$$\frac{a_1-a_{-1}}{a_0}=3\left[\tfrac{9}{8}-\frac{4-7m}{6-4m+m^2}\right]m^2,$$

$$\frac{a_2+a_{-2}}{a_0}=\tfrac{3}{16}\left[\tfrac{243}{16}+\frac{323+109m}{6-4m+m^2}-96\frac{23-11m}{[6-4m+m^2]^2}-\frac{215-53m}{30-4m+m^2}\right]m^4,$$

$$\frac{a_2-a_{-2}}{a_0}=\tfrac{3}{32}\left[\tfrac{243}{8}+\frac{175+563m}{6-4m+m^2}-48\frac{89-32m}{[6-4m+m^2]^2}+5\frac{361-10m}{30-4m+m^2}\right]m^4.$$

The evident objection to this form for the coefficients is that it makes the several terms very large, and of signs such that they nearly neutralize each other, the sum being very much smaller than any of the component terms. However it may be possible to remedy this imperfection by admitting three terms into the numerators, but, in this way, the problem is indeterminate, infinite variety being possible.

It is remarkable that none of our system of divisors can vanish for any real value of m, since the quadratic equations, obtained by equating them to zero, have all imaginary roots. In this they differ from the binomial divisors met with when the integration is effected in approximations arranged according to ascending powers of the disturbing force.

It is well known that the infinite series, obtained from the development, in ascending powers of m, of any fraction whose numerator is an entire function of m, and its denominator any integral power of a divisor of the previously mentioned series, is convergent, provided that m lies between the two square roots of the absolute term of the divisor. Hence any finite expression in m, involving these divisors, can be developed in such a series, provided that the numerical value of this parameter is less than $\sqrt{6}$. The same, however, cannot be asserted when the expression really forms an infinite series, as it is in the equation just given for the value of $\frac{a_i}{a_0}$. Yet, on account of the simplicity with which these quantities can be expressed in this form, a_1 and a_{-1} containing each a single term, with an error of the sixth order only, this limit is worthy of attention.

If the parameter m, hitherto employed by the lunar theorists, is taken as the quantity in powers of which to expand the value of a_i, we shall have $\mathrm{m} = \frac{m}{1 - m}$. And, substituting this value, the principal divisor $6 - 4\mathrm{m} + \mathrm{m}^2$ becomes $6 - 16m + 11m^2$. Thus the limits, between which m must be contained, in order that convergent series may be obtained where this divisor intervenes, are $\pm\sqrt{\tfrac{6}{11}}$. When we consider how little, in the case of our moon, m exceeds m, it will be plain that the series, in terms of m, are likely to be much more convergent than those in terms of m.

If we inquire what function of m, of the form $\frac{\mathrm{m}}{1 + \alpha\mathrm{m}}$, the quantity

$$\frac{M}{[6 - 4\mathrm{m} + \mathrm{m}^2]^k},$$

can be expanded in powers of, with the greatest convergency, it is easily found that α should be $-\frac{1}{3}$. Then putting

$$\mathrm{m} = \frac{\mathrm{m}}{1 + \frac{1}{3}\mathrm{m}},$$

the divisor $6 - 4m + m^2$ is changed into

$$6 + \tfrac{1}{3}\mathfrak{m}^2,$$

and there is introduced the additional divisor $1 + \frac{1}{3}\mathfrak{m}$. Here the series will be convergent provided $\mathfrak{m}$ is less than 3. It is true the terms involving the succeeding divisors $30 - 4m + m^2$, &c., are not benefited by this change of parameter, but as they play an inferior rôle in this matter, I have chosen $\mathfrak{m}$ as the parameter for the developments of the coefficients a_i in series of ascending powers.

To illustrate this matter, we have, in terms of the parameter $\mathfrak{m}$, and with errors of the sixth order,

$$\frac{a_1 + a_{-1}}{a_0} = -\left[\frac{2 + \frac{1}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{1}{1 + \frac{1}{3}\mathfrak{m}}\right]\mathfrak{m}^2,$$

$$\frac{a_1 - a_{-1}}{a_0} = \left[\frac{5 + \frac{7}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{7}{1 + \frac{1}{3}\mathfrak{m}} + \frac{\frac{27}{8}}{[1 + \frac{1}{3}\mathfrak{m}]^2}\right]\mathfrak{m}^2.$$

Expanding these expressions in powers of $\mathfrak{m}$, we get

$$\frac{a_1 + a_{-1}}{a_0} = -[\mathfrak{m}^2 + \tfrac{1}{2}\mathfrak{m}^3 - \tfrac{2}{9}\mathfrak{m}^4 + \tfrac{1}{36}\mathfrak{m}^5 + \ldots.],$$

$$\frac{a_1 - a_{-1}}{a_0} = \tfrac{11}{8}\mathfrak{m}^2 + \tfrac{5}{4}\mathfrak{m}^3 + \tfrac{5}{72}\mathfrak{m}^4 - \tfrac{11}{36}\mathfrak{m}^5 + \ldots.$$

Let these series be compared with those which correspond to them in the lunar theories of Plana or Delaunay, viz:

$$m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4 + \tfrac{383}{27}m^5 + \ldots.,$$

$$\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4 + \tfrac{2855}{108}m^5 + \ldots.$$

The superiority of the former, in convergence and simplicity of numerial coefficients, is manifest.

Much more might be said relative to possible modes of developing the coefficients a_i in series, but we content ourselves with giving their values expanded in powers of m, the series being carried to terms of the ninth order inclusive. The denominators of the numerical fractions are written as products of their prime factors, as, in this form, they can be more readily used, the principal labor in performing operations on these series being the reduc-

tion of the several fractional coefficients, to be added together, to a common denominator.

$$\frac{\mathrm{a}_1}{\mathrm{a}_0} = \frac{3}{2^4}\mathrm{m}^2 + \frac{1}{2}\mathrm{m}^3 + \frac{7}{2^2.3}\mathrm{m}^4 + \frac{11}{2^2.3^2}\mathrm{m}^5 - \frac{30749}{2^{12}.3^3}\mathrm{m}^6 - \frac{1010521}{2^{11}.3^4.5}\mathrm{m}^7 - \frac{18445871}{2^{10}.3^5.5^2}\mathrm{m}^8 - \frac{2114557853}{2^{12}.3^6.5^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-1}}{\mathrm{a}_0} = -\frac{19}{2^4}\mathrm{m}^2 - \frac{5}{3}\mathrm{m}^3 - \frac{43}{2^2.3^2}\mathrm{m}^4 - \frac{14}{3^3}\mathrm{m}^5 - \frac{7381}{2^{10}.3^4}\mathrm{m}^6 + \frac{3574153}{2^{11}.3^5.5}\mathrm{m}^7 + \frac{55218889}{2^9.3^6.5^2}\mathrm{m}^8 + \frac{13620153029}{2^{12}.3^7.5^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_2}{\mathrm{a}_0} = \frac{25}{2^8}\mathrm{m}^4 + \frac{803}{2^7.3.5}\mathrm{m}^5 + \frac{6109}{2^5.3^2.5^3}\mathrm{m}^6 + \frac{897599}{2^8.3^3.5^2}\mathrm{m}^7 + \frac{237203647}{2^{16}.3^2.5^4}\mathrm{m}^8 - \frac{44461407673}{2^{15}.3^4.5^5.7}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-2}}{\mathrm{a}_0} = 0\mathrm{m}^4 + \frac{23}{2^7.5}\mathrm{m}^5 + \frac{299}{2^5.3.5^2}\mathrm{m}^6 + \frac{56339}{2^8.3^2.5^3}\mathrm{m}^7 + \frac{79400351}{2^{16}.3^2.5^4}\mathrm{m}^8 + \frac{8085846833}{2^{14}.3^4.5^5.7}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_3}{\mathrm{a}_0} = \frac{833}{2^{12}.3}\mathrm{m}^6 + \frac{27943}{2^{11}.5.7}\mathrm{m}^7 + \frac{12275527}{2^{10}.3^2.5^2.7^2}\mathrm{m}^8 + \frac{27409853579}{2^{12}.3^4.5^3.7^3}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-3}}{\mathrm{a}_0} = \frac{1}{2^6.3}\mathrm{m}^6 + \frac{71}{2^7.3.5}\mathrm{m}^7 + \frac{46951}{2^8.3^2.5^2.7}\mathrm{m}^8 + \frac{14086643}{2^7.3^4.5^3.7^2}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_4}{\mathrm{a}_0} = \frac{3537}{2^{18}}\mathrm{m}^8 + \frac{111809667}{2^{17}.3^2.5.7^2}\mathrm{m}^9 \ldots$$

$$\frac{\mathrm{a}_{-4}}{\mathrm{a}_0} = \frac{23}{2^{11}.3}\mathrm{m}^6 + \frac{1576553}{2^{17}.3^2.7^2}\mathrm{m}^9 \ldots$$

These values being substituted in the equations

$$r\cos\upsilon = \Sigma_i \,.\, \mathrm{a}_i \cos 2i\tau\,,$$
$$r\sin\upsilon = \Sigma \,.\, \mathrm{a}_i \sin 2i\tau\,,$$

and the parameter changed to $\mathfrak{m}$, we get

$$\begin{aligned} r\cos\upsilon = \mathrm{a}_0 \Big\{ 1 &+ \Big[-\mathfrak{m}^2 - \frac{1}{2}\mathfrak{m}^3 + \frac{2}{9}\mathfrak{m}^4 - \frac{1}{36}\mathfrak{m}^5 - \frac{106411}{331776}\mathfrak{m}^6 + \frac{427339}{497664}\mathfrak{m}^7 \\ &+ \frac{25239037}{14929920}\mathfrak{m}^8 - \frac{732931}{37324800}\mathfrak{m}^9 \ldots \Big] \cos 2\tau \\ &+ \Big[\frac{25}{256}\mathfrak{m}^4 + \frac{311}{960}\mathfrak{m}^5 + \frac{9349}{28800}\mathfrak{m}^6 - \frac{5831}{216000}\mathfrak{m}^7 \\ &- \frac{164645363}{552960000}\mathfrak{m}^8 - \frac{11321875589}{19353600000}\mathfrak{m}^9 \ldots \Big] \cos 4\tau \\ &+ \Big[\frac{299}{4096}\mathfrak{m}^6 + \frac{30193}{107520}\mathfrak{m}^7 + \frac{379549}{1003520}\mathfrak{m}^8 + \frac{181908179}{1580544000}\mathfrak{m}^9 \ldots \Big] \cos 6\tau \\ &+ \Big[\frac{11347}{196608}\mathfrak{m}^8 + \frac{2350381}{9031680}\mathfrak{m}^9 \ldots \Big] \cos 8\tau + \ldots \Big\}, \end{aligned}$$

$$
\begin{aligned}
r\sin\upsilon = \mathrm{a}_0 \Big\{ & \left[\frac{11}{8}\mathrm{m}^2 + \frac{5}{4}\mathrm{m}^3 + \frac{5}{72}\mathrm{m}^4 - \frac{11}{36}\mathrm{m}^5 - \frac{101123}{331776}\mathrm{m}^6 - \frac{512239}{276480}\mathrm{m}^7 \right. \\
& \left. - \frac{269023019}{74649600}\mathrm{m}^8 - \frac{151872119}{93312000}\mathrm{m}^9 \ldots \right] \sin 2\tau \\
& + \left[\frac{25}{256}\mathrm{m}^4 + \frac{121}{480}\mathrm{m}^5 + \frac{5623}{28800}\mathrm{m}^6 - \frac{17149}{432000}\mathrm{m}^7 \right. \\
& \left. - \frac{3500287}{11520000}\mathrm{m}^8 - \frac{43885512859}{58060800000}\mathrm{m}^9 \ldots \right] \sin 4\tau \\
& + \left[\frac{769}{12288}\mathrm{m}^6 + \frac{24481}{107520}\mathrm{m}^7 + \frac{4419347}{15052800}\mathrm{m}^8 + \frac{398314169}{4741632000}\mathrm{m}^9 \ldots \right] \sin 6\tau \\
& + \left[\frac{9875}{196608}\mathrm{m}^8 + \frac{32608451}{144506880}\mathrm{m}^9 \ldots \right] \sin 8\tau + \ldots \Big\}.
\end{aligned}
$$

Our final differential equations are capable of furnishing only the ratios of the coefficients a_i, hence we must have recourse to one of the original equations if we wish to determine a_0 as a function of n and μ. By substituting the values

$$u = \Sigma_i \,.\, \mathrm{a}_i \zeta^{2i+1}, \qquad s = \Sigma_i \,.\, \mathrm{a}_{-i-1} \zeta^{2i+1},$$

in the differential equation

$$\left[D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right] u + \tfrac{3}{2}\mathrm{m}^2 s = 0,$$

we obtain

$$\frac{\varkappa u}{(us)^{\frac{3}{2}}} = \Sigma_i \,.\, \{[(2i + 1 + \mathrm{m})^2 + \tfrac{1}{2}\mathrm{m}^2]\,\mathrm{a}_i + \tfrac{3}{2}\mathrm{m}^2 \mathrm{a}_{-i-1}\}\, \zeta^{2i+1}.$$

Considering only the term of this, for which $i = 0$, and supposing that the coefficient of ζ in the expansion of $\frac{\mathrm{a}_0^2 u}{(us)^{\frac{3}{2}}}$ is denoted by J, we shall have

$$\frac{\varkappa}{\mathrm{a}_0^3} J = 1 + 2\mathrm{m} + \tfrac{3}{2}\mathrm{m}^2 + \tfrac{3}{2}\mathrm{m}^2 \frac{\mathrm{a}_{-1}}{\mathrm{a}_0}.$$

For brevity call the right member of this H; then, since

$$\varkappa = \frac{\mu}{(n - n')^2} = \frac{\mu}{n^2}(1 + \mathrm{m})^2,$$

we shall have

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{J(1 + \mathrm{m})^2}{H}\right]^{\frac{1}{3}}.$$

The value of H is readily obtained from the value of $\frac{\mathrm{a}_{-1}}{\mathrm{a}_0}$ given above, and J must be found by substituting the values

$$u = \Sigma_i \,.\, \mathrm{a}_i \zeta^{2i+1}, \qquad s = \Sigma_i \,.\, \mathrm{a}_{-i-1} \zeta^{2i+1},$$

in $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$, and taking the coefficient of ζ. We get

$$J = 1 + \left[\frac{a_1 + a_{-1}}{a_0}\right]^2 \left[\tfrac{3}{4} + \tfrac{45}{64}\left[\frac{a_1 + a_{-1}}{a_0}\right]^2 + \tfrac{15}{8}\frac{a_1 a_{-1}}{a_0^2} - \tfrac{15}{2}\frac{a_2 + a_{-2}}{a_0}\right]$$
$$+ \frac{a_2 + a_{-2}}{a_0}\left[\tfrac{3}{4}\frac{a_2 + a_{-2}}{a_0} + 6\frac{a_1 a_{-1}}{a_0^2}\right] + 6\frac{a_1 + a_{-1}}{a_0}\frac{a_1 a_2 + a_{-1}a_{-2}}{a_0^2}$$
$$+ 3\frac{a_1 a_{-1}}{a_0^2} + 45\frac{a_1^2 a_{-1}^2}{a_0^4} + 3\frac{a_2 a_{-2}}{a_0^2},$$

where the terms neglected are, at lowest, of the tenth order with respect to m. And, explicitly in terms of this parameter,

$$J = 1 + \frac{21}{2^8}m^4 - \frac{31}{2^5}m^5 - \frac{53}{2^4}m^6 - \frac{2707}{2^6.3^2}m^7 - \frac{4201213}{2^{10}.3^3}m^8 + \frac{14374939}{2^{13}.3^3.5}m^9 \ldots$$

By means of which there is obtained

$$a_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}\left[1 - \frac{1}{6}m^2 + \frac{1}{3}m^3 + \frac{407}{2304}m^4 - \frac{67}{288}m^5 - \frac{45293}{41472}m^6 \right.$$
$$\left. - \frac{8761}{6912}m^7 - \frac{4967441}{7962624}m^8 + \frac{14829273}{39813120}m^9 \ldots\right],$$

or, in terms of the parameter $\mathfrak{m}$,

$$a_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}\left[1 - \frac{1}{6}\mathfrak{m}^2 + \frac{4}{9}\mathfrak{m}^3 - \frac{163}{768}\mathfrak{m}^4 - \frac{1147}{5184}\mathfrak{m}^5 - \frac{79859}{124416}\mathfrak{m}^6 \right.$$
$$\left. + \frac{4811}{10368}\mathfrak{m}^7 + \frac{9520295}{71663616}\mathfrak{m}^8 + \frac{139240651}{1074954240}\mathfrak{m}^9 \ldots\right].$$

The quantity $\left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}$ is usually designated a by the lunar-theorists; and, to make this appear as a factor of the expressions for $r \cos v$ and $r \sin v$, it would be necessary to multiply all the coefficients by the second factor of the preceding expression for a_0. It seems simpler however to retain a_0 as the factor of linear magnitude; for the astronomers have preferred to derive the constant of lunar parallax from direct observation of the moon, or, in other words, they have preferred to consider μ as a seventh element of the orbit; with this view of the matter, there is no incongruity in making a_0 everywhere replace μ.

The expression for a_0 can be obtained in several other ways, which lead to more symmetrical formulæ, and which also serve for verification of all the preceding developments. If, in the preceding equation giving the value of $\frac{xu}{(us)^{\frac{3}{2}}}$ in terms of ζ, we attribute to τ the value 0, or, which is equivalent, make $\zeta = 1$, we shall have $u = s = \Sigma_i . a_i$, and, consequently

$$\frac{\varkappa}{[\Sigma_i \,.\, \mathrm{a}_i]^2} = \Sigma_i \,.\, [(2i+1+\mathrm{m})^2 + 2\mathrm{m}^2]\,\mathrm{a}_i .$$

And thus, mindful of the value of $\varkappa$ given above, we get

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{(1+\mathrm{m})^2}{\Sigma_i \,.\, [(2i+1+\mathrm{m})^2 + 2\mathrm{m}^2]\,\frac{\mathrm{a}_i}{\mathrm{a}_0} \,.\, \left[\Sigma_i \,.\, \frac{\mathrm{a}_i}{\mathrm{a}_0}\right]^2}\right]^{\frac{1}{3}} .$$

Again the differential equation

$$\frac{d^2y}{d\tau^2} + 2\mathrm{m}\,\frac{dx}{d\tau} + \frac{\varkappa}{r^3}\,y = 0$$

gives

$$\frac{\varkappa}{r^3} \,.\, y = \Sigma_i \,.\, [(2i+1+\mathrm{m})^2 - \mathrm{m}^2]\,\mathrm{a}_i \sin(2i+1)\tau ,$$

and, attributing to τ the special value $\frac{\pi}{2}$,

$$\frac{\varkappa}{[\Sigma_i \,.\, (-1)^i \mathrm{a}_i]^2} = \Sigma_i \,.\, (-1)^i (2i+1)(2i+1+\mathrm{m})\,\mathrm{a}_i .$$

Whence

$$\mathrm{a}_0 = \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}} \left[\frac{(1+\mathrm{m})^2}{\Sigma_i \,.\, (-1)^i (2i+1)(2i+1+\mathrm{m})\,\frac{\mathrm{a}_i}{\mathrm{a}_0} \,.\, \left[\Sigma_i \,.\, (-1)^i \frac{\mathrm{a}_i}{\mathrm{a}_0}\right]^2}\right]^{\frac{1}{3}} .$$

When $j=0$ in the first equation of condition for determining the coefficients a_i, we get a formula expressing C in terms of these quantities, viz.,

$$C = \Sigma_i \,.\, [(2i+1+2\mathrm{m})^2 + \tfrac{1}{2}\mathrm{m}^2]\,\mathrm{a}_i^2 + \tfrac{9}{2}\mathrm{m}^2 \Sigma_i \,.\, \mathrm{a}_i \mathrm{a}_{-i-1} ,$$

or neglecting terms of the eight and higher orders,

$$\begin{aligned} C &= \mathrm{a}_0^2 \left[1 + 4\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2 + (9 + 12\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2)\,\frac{\mathrm{a}_1^2}{\mathrm{a}_0^2} + (1 - 4\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2)\,\frac{\mathrm{a}_{-1}^2}{\mathrm{a}_0^2} + 9\mathrm{m}^2\,\frac{\mathrm{a}_{-1}}{\mathrm{a}}\right] \\ &= \mathrm{a}_0^2 \left[1 + 4\mathrm{m} + \tfrac{9}{2}\mathrm{m}^2 - \frac{1147}{2^7}\mathrm{m}^4 - \frac{1399}{2^5.3}\mathrm{m}^5 - \frac{2047}{2^8}\mathrm{m}^6 + \frac{3737}{2^4.3^3}\mathrm{m}^7\right] . \end{aligned}$$

But the C of Chap. I is obtained by multiplying this C by $\frac{1}{2}\nu^2 = \frac{1}{2}\,\frac{n^2}{(1+\mathrm{m})^2}$. Hence, substituting for a_0 its value, we have

$$C = \tfrac{1}{2}(\mu n)^{\frac{2}{3}} \left[1 + 2\mathrm{m} - \tfrac{5}{6}\mathrm{m}^2 - \mathrm{m}^3 - \frac{1319}{288}\mathrm{m}^4 - \frac{67}{144}\mathrm{m}^5 - \frac{2879}{1296}\mathrm{m}^6 - \frac{1321}{1296}\mathrm{m}^7\right] ,$$

as there stated.

We propose now to reduce the preceding formulæ to numerical results. For this purpose we assume

$$n = 17325594''.06085 ,$$
$$n' = 1295977''.41516 ,$$

which give

$$\begin{aligned}
m &= \frac{n'}{n-n'} = 0.08084\ 89338\ 08312\,, \\
m^2 &= 0.00653\ 65500\ 97941\,, \\
m^3 &= 0.00052\ 84731\ 06203\,, \\
m^4 &= 0.00004\ 27264\ 87183\,, \\
\mathfrak{m} &= 0.08308\ 81293\ 65\,.
\end{aligned}$$

The numerical value of $\mathfrak{m}$ being substituted in the series, we obtain

$$a_0 = 0.99909\ 31419\ 62 \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}},$$

$$\begin{aligned}
r\cos v = a_0 [1 &- 0.00718\ 00394\ 55 \cos 2\tau \\
&+ 0.00000\ 60424\ 59 \cos 4\tau \\
&+ 0.00000\ 00325\ 76 \cos 6\tau \\
&+ 0.00000\ 00001\ 80 \cos 8\tau], \\
r\sin v = a_0 [&\ 0.01021\ 14543\ 96 \sin 2\tau \\
&+ 0.00000\ 57148\ 79 \sin 4\tau \\
&+ 0.00000\ 00274\ 99 \sin 6\tau \\
&+ 0.00000\ 00001\ 57 \sin 8\tau].
\end{aligned}$$

The method of employing numerical values, from the outset, in the equations of condition, determining the a_i, is far less laborious than the literal development of these coefficients in powers of a parameter. For comparison with the results just given, we add the calculation of the coefficients by this method. The following table gives the numerical values of the symbols $[j, i]$, $[j]$ and (j), but the division by the quantity $2(4j^2-1)-4m+m^2$ has been omitted; it is easier to perform this once for all at the end of the series of operations, than to divide each coefficient separately. Hence it must be understood that all the numbers in each department of the table are to be divided by the divisor which stands at the head of it.

Coefficients for a_1 and a_{-1}.

Divisor = 5.68314 08148 64695.

$[1] =$	0.00861 47842 96261	$[-1] = -$	0.01178 75756 56865
$(1) = -$	0.00623 66553 18347	$(-1) = -$	0.04941 95042 02516
$[1, -2] =$	13.30665 60411	$[-1, -3] = -$	66.98979 68560
$[1, -1] =$	6.32993 22853	$[-1, -2] = -$	28.01307 31002
$[1, 2] = -$	10.71949 01593	$[-1, 1] = -$	10.96365 06556
$[1, 3] = -$	15.10904 80332	$[-1, 2] = -$	38.57409 27816

Coefficients for a_2 and a_{-2}.

Divisor = 29.68314 08148 64695.

$2[2] =$	0.00205 43632 76229	$2[-2] = -$	0.01834 79966 76898
$2(2) = -$	0.02909 07097 39048	$2(-2) = -$	0.07227 35586 23216
$[2,-2] =$	14.97672 37558	$[-2,-4] = -$	108.69586 45706
$[2,-1] =$	9.32666 40103	$[-2,-3] = -$	63.00980 48251
$[2,1] = -$	13.00326 82750 49	$[-2,-1] = -$	8.67987 25398
$[2,3] = -$	50.03961 76194	$[-2,1] = -$	3.64352 31954
$[2,4] = -$	74.07269 86888	$[-2,2] = -$	19.61044 21261

Coefficients for a_3 and a_{-3}.

Divisor = 69.68314 08149.

$[3] = -$	0.00113 35729 26473	$[-3] = -$	0.00793 43596
$(3) = -$	0.01768 33677	$(-3) = -$	0.03207 76506 69434
$[3,-1] =$	12.99224 12519	$[-3,-4] = -$	114.67538 20668
$[3,1] = -$	18.10997 74284	$[-3,-2] = -$	35.57316 33864
$[3,2] = -$	41.33769 10334	$[-3,-1] = -$	12.34544 97815
$[3,4] = -$	103.14632 67728	$[-3,1] =$	1.46318 59580

Coefficients for a_4 and a_{-4}.

Divisor = 125.68314 08.

$2[4] = -$	0.00428 9733	$2[-4] = -$	0.01449 0913
$2(4) = -$	0.03864 29156	$2(-4) = -$	0.06023 435
$[4,-1] =$	16.82502 987	$[-4,-5] = -$	182.50817 069
$[4,1] = -$	22.66333 2	$[-4,-3] = -$	79.01980 9
$[4,2] = -$	51.16496 6	$[-4,-2] = -$	42.51817 5
$[4,3] = -$	85.50490 2	$[-4,-1] = -$	16.17823 8
$[4,5] = -$	171.69968 135	$[-4,1] =$	6.01654 053

Coefficients for a_5 and a_{-5}.

Divisor = 197.68314.

$[5] = -$	0.00272 9536	$(5) = -$	0.02896 299
$[5,1] = -$	26.99534 4	$[-5,-4] = -$	138.68780 0
$[5,2] = -$	60.26133 2	$[-5,-3] = -$	89.42181 0
$[5,3] = -$	99.79795 8	$[-5,-2] = -$	49.88518 4
$[5,4] = -$	145.60523 2	$[-5,-1] = -$	20.07791 2

Coefficients for a_6 and a_{-6}.

Divisor = 285.68314.

$2[6] = -$	0.00622 021	$2(-6) = -$	0.05640 548
$[6,1] = -$	31.21669	$[-6,-5] = -$	214.46646
$[6,2] = -$	68.99224	$[-6,-4] = -$	152.69091
$[6,3] = -$	113.32666	$[-6,-3] = -$	100.35648
$[6,4] = -$	164.21995	$[-6,-2] = -$	57.46319
$[6,5] = -$	221.67212	$[-6,-1] = -$	24.01103.

These numbers are arranged for carrying the precision to quantities of the 13[th] order inclusive, and to 15 places of decimals. The quantities $[j, i]$ can be tested by differences, if 0 and the divisor with the negative sign are inserted in the proper places in the series of numbers; for it is evident that the second differences should be constant.

The final results are given below, where, in order that the degree of convergence of this process may be appreciated, we have given the value arising from the first approximation, and then, separately, the corrections arising severally from the second and third approximations. It must be borne in mind that each of these terms is the numerical value, not of an infinite series, but of a rational function of m, and, consequently, admits of being computed exact to the last decimal place employed, and, in fact, is here so computed. Hence any error there may be in these values of the a_i arises only from the neglect of the terms of the following approximations, which, in half the number of cases, are of the 14[th] order, and, in the other half, of the 16[th] order. It is safe to affirm that these cannot, in any case, exceed two units in the 15[th] decimal.

	a_1.	a_{-1}.
1st apx., term of 2d order,	+ 0.00151 58491 71593	− 0.00869 58084 99634
2d " " 6th "	− 0.00000 01416 98831	+ 0.00000 00615 51932
3d " " 10th "	+ 0.00000 00000 06801	− 0.00000 00000 13838
	$\frac{a_1}{a_0}$ = + 0.00151 57074 79563,	$\frac{a_{-1}}{a_0}$ = − 0.00869 57469 61540,

	a_2.	a_{-2}.
1st apx., term of 4th order,	+ 0.00000 58793 35016	+ 0.00000 01636 69405
2d " " 8th "	− 0.00000 00006 78490	+ 0.00000 00001 21088
3d " " 12th "	+ 0.00000 00000 00052	− 0.00000 00000 00007
	$\frac{a_2}{a_0}$ = + 0.00000 58786 56578,	$\frac{a_{-2}}{a_0}$ = + 0.00000 01637 90486,

	a_3.	a_{-3}.
1st apx., term of 6th order,	+ 0.00000 00300 35759	+ 0.00000 00024 60338
2d " " 10th "	− 0.00000 00000 04128	+ 0.00000 00000 00055
	$\frac{a_3}{a_0}$ = + 0.00000 00300 31632,	$\frac{a_{-3}}{a_0}$ = + 0.00000 00024 60393,

	a_4.	a_{-4}.
1st apx., term of 8th order,	+ 0.00000 00001 75296	+ 0.00000 00000 12284
2d " " 12th "	− 0.00000 00000 00028	0.00000 00000 00000
	$\frac{a_4}{a_0}$ = + 0.00000 00001 75268,	$\frac{a_{-4}}{a_0}$ = + 0.00000 00000 12284,

Of the 10th order, $\frac{a_5}{a_0} = + 0.00000\ 00000\ 01107$, $\frac{a_{-5}}{a_0} = + 0.00000\ 00000\ 00064$,

Of the 12th order, $\frac{a_6}{a_0} = + 0.00000\ 00000\ 00007$, $\frac{a_{-6}}{a_0} = + 0.00000\ 00000\ 00000$.

These give the following numerical expression for the coordinates:

$$
\begin{aligned}
r \cos \upsilon = a_0 \,[1 & - 0.00718\ 00394\ 81977 \cos 2\tau \\
& + 0.00000\ 60424\ 47064 \cos 4\tau \\
& + 0.00000\ 00324\ 92024 \cos 6\tau \\
& + 0.00000\ 00001\ 87552 \cos 8\tau \\
& + 0.00000\ 00000\ 01171 \cos 10\tau \\
& + 0.00000\ 00000\ 00008 \cos 12\tau], \\
r \sin \upsilon = a_0 \,[& \ 0.01021\ 14544\ 41102 \sin 2\tau \\
& + 0.00000\ 57148\ 66093 \sin 4\tau \\
& + 0.00000\ 00275\ 71239 \sin 6\tau \\
& + 0.00000\ 00001\ 62985 \sin 8\tau \\
& + 0.00000\ 00000\ 01042 \sin 10\tau \\
& + 0.00000\ 00000\ 00007 \sin 12\tau].
\end{aligned}
$$

On comparison of these values with those obtained from the series in m, the differences are found to be only some units in the 11th decimal.

The coefficients tend to diminish with some regularity as we advance towards higher orders. This is shown by the following scheme of the logarithms and their differences:

	Δ	Δ^2		Δ	Δ^2
n 97.8561			98.0091		
				— 3.2521	
94.7812			94.7570		+ 9356
	— 2.2694			2.3165	
92.5118		+ 307	92.4405		871
	2.2387			2.2294	
90.2731		341	90.2111		363
	2.2046			2.1931	
88.0685		237	88.0180		201
	2.1809			2.1730	
85.8876			85.8450		

For verification, the following equations were computed:

$$\Sigma_i \,.\, [(2i + 1 + m)^2 + 2m^2]\, a_i \,.\, [\Sigma_i \,.\, a_i]^2 = 1.17141\ 84591\ 84518 a_0^3 .$$
$$\Sigma_i \,.\, (-1)^i (2i + 1)(2i + 1 + m)\, a_i \,.\, [\Sigma_i \,.\, (-1)^i a_i]^2 = 1.17141\ 84591\ 84513 a_0^3 .$$

The small difference between the numbers is explained by the fact that, in these formulæ, the quantities a_i are, when i is somewhat large, multiplied by large numbers; as, for instance, a_6 by 169. From the average of these two results, we get

$$a_0 = 0.99909\ 31419\ 75298 \left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}.$$

In the investigations of succeeding chapters, the function $\frac{x}{r^3}$ plays an important part. Hence we will here derive its development as a periodic function of τ by the method of special values. By dividing the quadrant, with reference to τ, into 6 equal parts, we obtain the advantage that the sines or cosines of the multiples of 2τ are either rational or involve $\sqrt{3}$. The special values of the coordinates and of $\frac{x}{r^3}$, thence deduced, are

τ.	$\frac{r}{a_0}\cos v$.	$\frac{r}{a_0}\sin v$.	$\frac{x}{r^3}$.
0°	0.99282 60356 45842	0.00000 00000 00000	1.19699 57017 23421
15	0.99378 49245 37167	0.00511 07041 52675	1.19348 68051 03032
30	0.99640 69264 50272	0.00884 83280 32746	1.18399 66676 76716
45	0.99999 39577 40480	0.01021 14268 70906	1.17125 64904 33157
60	1.00358 70309 15127	0.00883 84298 76613	1.15876 77987 29687
75	1.00622 11177 22330	0.00510 08054 31947	1.14978 07679 95764
90	1.00718 60496 23406	0.00000 00000 00000	1.14652 34925 50570.

From the numbers of the last column, by the known process, we deduce

$$\begin{aligned}\frac{x}{r^3} = \;& 1.17150\,80211\,79225 \\ &+0.02523\,36924\,97860 \cos 2\tau \\ &+0.00025\,15533\,50012 \cos 4\tau \\ &+0.00000\,24118\,79799 \cos 6\tau \\ &+0.00000\,00226\,05851 \cos 8\tau \\ &+0.00000\,00002\,08750 \cos 10\tau \\ &+0.00000\,00000\,01908 \cos 12\tau \\ &+0.00000\,00000\,00017 \cos 14\tau.\end{aligned}$$

The last coefficient has been added from induction, after which it becomes necessary, as is plain, to subtract an equal quantity from the coefficient of cos 10τ. Writing the logarithms, as in the former case, we have, the last logarithm being supplied from estimation,

	Δ	Δ^2	Δ^3
98.4020			
	—2.0014		
96.4006		—168	
	2.0182		+68
94.3824		100	
	2.0282		36
92.3542		64	
	2.0346		20
90.3196		44	
	2.0390		10
88.2806		34	
	2.0424		
86.2382			

It will be noticed how much slower this series converges than those for the coordinates.

Any information regarding the motion of satellites having long periods of revolution about their primaries will doubtless be welcome, as the series given by previous investigators are inadequate for showing anything in this direction. Hence this chapter will be terminated by a table of the more salient properties of the class of satellites having the radius vector at a minimum in syzygies and at a maximum in quadratures. For this end I have selected, besides the earth's moon, taken for the sake of comparison, the moons of 10, 9, 8, ... , 3 lunations in the periods of their primaries, and also what may be called the moon of maximum lunation, as, of the class of satellites under discussion, exhibiting the complete round of phases, it has the longest lunation.*

In order that the table may be readily applicable to satellites accompanying any planet, the canonical linear and temporal units, that is those for which μ and n' are both unity, will be used.

From the foregoing methods we obtain:

$$\text{For } \mathrm{m} = \frac{1}{10};$$

$$r \cos v = \mathrm{a}\,[1 - 0.011230 \cos 2\tau + 0.000015 \cos 4\tau], \qquad r \sin v = \mathrm{a}\,[\,0.016102 \sin 2\tau + 0.000014 \sin 4\tau],$$

$$\log \mathrm{a} = 9.3051648.$$

$$\text{For } \mathrm{m} = \frac{1}{9};$$

$$r \cos v = \mathrm{a}\,[1 - 0.014044 \cos 2\tau + 0.0000247 \cos 4\tau], \qquad r \sin v = \mathrm{a}\,[\,0.020232 \sin 2\tau + 0.0000230 \sin 4\tau],$$

$$\log \mathrm{a} = 9.3326467.$$

$$\text{For } \mathrm{m} = \frac{1}{8};$$

$$r \cos v = \mathrm{a}\,[1 - 0.018061 \cos 2\tau + 0.0000421 \cos 4\tau + 0.00000057 \cos 6\tau], \qquad r \sin v = \mathrm{a}\,[\,0.026172 \sin 2\tau + 0.0000388 \sin 4\tau + 0.00000048 \sin 6\tau],$$

$$\log \mathrm{a} = 9.3630019.$$

$$\text{For } \mathrm{m} = \frac{1}{7};$$

$$r \cos v = \mathrm{a}\,[1 - 0.02407886 \cos 2\tau + 0.00007760 \cos 4\tau + 0.00000141 \cos 6\tau + 0.000000025 \cos 8\tau],$$

$$r \sin v = \mathrm{a}\,[\,0.03516059 \sin 2\tau + 0.00007063 \sin 4\tau + 0.00000118 \sin 6\tau + 0.000000022 \sin 8\tau],$$

$$\log \mathrm{a} = 9.3969048.$$

* The attribution of the maximum lunation to this moon is erroneous as was first pointed out to me by J. C. Adams and afterwards by M. Poincaré.

For $m = \frac{1}{6}$;

$$r \cos \upsilon = a\,[1 - 0.03368245 \cos 2\tau + 0.00015943 \cos 4\tau + 0.000004077 \cos 6\tau + 0.000000097 \cos 8\tau],$$

$$r \sin \upsilon = a\,[\ 0.04968194 \sin 2\tau + 0.00014312 \sin 4\tau + 0.000003393 \sin 6\tau + 0.000000084 \sin 8\tau],$$

$$\log a = 9.4352928.$$

For $m = \frac{1}{5}$;

$$r \cos \upsilon = a\,[1 - 0.05038803 \cos 2\tau + 0.00038127 \cos 4\tau + 0.000014686 \cos 6\tau + 0.000000505 \cos 8\tau],$$

$$r \sin \upsilon = a\,[\ 0.07536021 \sin 2\tau + 0.00033582 \sin 4\tau + 0.000012168 \sin 6\tau + 0.000000438 \sin 8\tau],$$

$$\log a = 9.4795445.$$

For $m = \frac{1}{4}$;

$$r \cos \upsilon = a\,[1 - 0.08331972 \cos 2\tau + 0.00114564 \cos 4\tau + 0.00007409 \cos 6\tau + 0.00000404 \cos 8\tau],$$

$$r \sin \upsilon = a\,[\ 0.12709553 \sin 2\tau + 0.00098090 \sin 4\tau + 0.00006099 \sin 6\tau + 0.00000342 \sin 8\tau].$$

$$\log a = 9.5318013.$$

For $m = \frac{1}{3}$;

$$r \cos \upsilon = a\,[1 - 0.1622330 \cos 2\tau + 0.0048920 \cos 4\tau + 0.00059858 \cos 6\tau + 0.000081198 \cos 8\tau + 0.000011873 \cos 10\tau + 0.000001849 \cos 12\tau],$$

$$r \sin \upsilon = a\,[\ 0.2542740 \sin 2\tau + 0.0039840 \sin 4\tau + 0.00049306 \sin 6\tau + 0.000070196 \sin 8\tau + 0.000010611 \sin 10\tau + 0.0000016902 \sin 12\tau],$$

$$\log a = 9.5955815.$$

For moons of much longer lunations the methods hitherto used are not practicable, and, in consequence, we resort to mechanical quadratures. Here we shall have two cases. The satellite may be started at right angles to and from a point on the line of syzygies, and the motion traced across the first quadrant; or it may be started at right angles to and from a point on the line of quadratures, and the motion traced across the second quadrant; the prime object being to discover what value of the initial velocity will make the satellite intersect perpendicularly the axis at the farther side of the quadrant.

The differential equations

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} + \left[\frac{1}{r^3} - 3\right]x = 0,$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + \frac{y}{r^3} \qquad\qquad = 0,$$

give, as expressions of the values of the coordinates, in the first case,

$$x = x_0 + 2\int_0^t y dt - \int_0^t \int_0^t \left[\frac{1}{r^3} - 3\right] x dt^2,$$

$$y = 2\int_0^t (x_0 - x)\, dt - \int_0^t \int_0^t \frac{y}{r^3} dt^2,$$

and, in the second case,

$$x = -2\int_0^t (y_0 - y)\, dt - \int_0^t \int_0^t \left[\frac{1}{r^3} - 3\right] x dt^2,$$

$$y = y_0 - 2\int_0 x dt - \int_0 \int_0^t \frac{y}{r^3} dt^2.$$

Here the subscript ($_0$) denotes values which belong to the beginning of motion, and ($_1$) will hereafter be used to denote those which belong to the end.

Let v be the velocity, and σ the angle, the direction of motion, relative to the rotating axes, makes with the moving line of syzygies. In the first case then $\sigma_0 = 90°$, and we wish to ascertain what value of v_0 will make $\sigma_1 = 180°$. Generally, for small values of v_0, σ_1 will come out but little less than 270°; but, as v_0 augments, σ_1 will be found to diminish, and, if x_0 does not exceed a certain limit, a value of v_0 can be found which will make $\sigma_1 = 180°$. In the second case, in like manner, we seek what value of v_0 will make $\sigma_1 = 270°$.

Mechanical quadratures performed with axes of coordinates having no rotation possess some advantages, as, in this case, the velocities are not present in the expressions of the second differentials of the coordinates.

Let X and Y denote the coordinates of the moon in this system, and λ its longitude measured from the line of the last syzygy, from which t is also counted. Then the potential function is

$$\Omega = \frac{1}{r} - \frac{1}{2} r^2 + \frac{3}{2}(X \cos t + Y \sin t)^2.$$

And

$$\frac{d^2X}{dt^2} = \frac{d\Omega}{dX} = -\left[\frac{1}{r^3} + 1\right] X + 3r \cos(\lambda - t) \cos t,$$

$$\frac{d^2Y}{dt^2} = \frac{d\Omega}{dY} = -\left[\frac{1}{r^3} + 1\right] Y + 3r \cos(\lambda - t) \sin t.$$

Therefore, if we compute p and θ from

$$p\cos\theta = -\left[\frac{1}{r^3} - 2r\right]\cos(\lambda - t),$$

$$p\sin\theta = -\left[\frac{1}{r^3} + r\right]\sin(\lambda - t),$$

we shall have

$$\frac{d^2X}{dt^2} = p\cos(\theta + t),$$

$$\frac{d^2Y}{dt^2} = p\sin(\theta + t).$$

The needed values of v and σ can be derived from the equations

$$v\cos(\sigma + t) = \frac{dX}{dt} + Y,$$

$$v\sin(\sigma + t) = \frac{dY}{dt} - X.$$

The developments of the coordinates in ascending powers of t, t being counted from any desired epoch, can often be employed with advantage. Differentiating the differential equations n times we have

$$\frac{d^{n+2}x}{dt^{n+2}} = 2\frac{d^{n+1}y}{dt^{n+1}} + 3\frac{d^nx}{dt^n} - \frac{d^n}{dt^n}(r^{-3}x),$$

$$\frac{d^{n+2}y}{dt^{n+2}} = -2\frac{d^{n+1}x}{dt^{n+1}} - \frac{d^n}{dt^n}(r^{-3}y).$$

Also

$$\frac{d^n}{dt^n}(r^{-3}x) = r^{-3}\frac{d^nx}{dt^n} + n\frac{d(r^{-3})}{dt}\frac{d^{n-1}x}{dt^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^2(r^{-3})}{dt^2}\frac{d^{n-2}x}{dt^{n-2}} + \ldots,$$

with a similar formula for the differential coefficients of $r^{-3}y$.

The differential coefficients of r^{-3}, as far as the 4th, are

$$\frac{d(r^{-3})}{dt} = -\quad 3r^{-5}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right),$$

$$\frac{d^2(r^{-3})}{dt^2} = -\quad 3r^{-5}\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) + 15r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2,$$

$$\begin{aligned}\frac{d^3(r^{-3})}{dt^3} = -\quad & 3r^{-5}\left(x\frac{d^3x}{dt^3} + y\frac{d^3y}{dt^3} + 3\frac{dx}{dt}\frac{d^2x}{dt^2} + 3\frac{dy}{dt}\frac{d^2y}{dt^2}\right) \\ & + 45r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) \\ & - 105r^{-9}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^3,\end{aligned}$$

$$
\begin{aligned}
\frac{d^4(r^{-3})}{dt^4} = &- 3r^{-5}\left[x\frac{d^4x}{dt^4} + y\frac{d^4y}{dt^4} + 4\frac{dx}{dt}\frac{d^3x}{dt^3} + 4\frac{dy}{dt}\frac{d^3y}{dt^3} + 3\left(\frac{d^2x}{dt^2}\right)^2 + 3\left(\frac{d^2y}{dt^2}\right)^2\right] \\
&+ 60r^{-7}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)\left(x\frac{d^3x}{dt^3} + y\frac{d^3y}{dt^3} + 3\frac{dx}{dt}\frac{d^2x}{dt^2} + 3\frac{dy}{dt}\frac{d^2y}{dt^2}\right) \\
&+ 45r^{-7}\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)^2 \\
&- 630r^{-9}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2\left(x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) \\
&+ 945r^{-11}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^4.
\end{aligned}
$$

By means of these formulæ x and y can be expanded in series of ascending powers of t, as far as the term involving t^6, provided we know the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ corresponding to $t = 0$. Taking t sufficiently small to make the terms, involving higher powers of t than the sixth, insignificant, as, for instance, $t = 0.05$ or $t = 0.1$, we can ascertain the values of x, y, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the end of this time. With these values we can again construct new series for x and y in powers of t, in which the latter variable is counted from the end of the previous time. By repetitions of this process the integration can be carried as far as desired. Jacobi's integral, which has not been put to use in the preceding formulæ, can be employed as a check.

In case the body starts from, and at right angles to, either axis, the coefficients of every other power of t in the series for the coordinates vanish.

Thus when the axis in question is that of x, the series for the coordinates have the forms

$$
\begin{aligned}
x &= x_0 + A_2t^2 + A_4t^4 + A_6t^6 + A_8t^8 + \ldots, \\
y &= v_0t + A_3t^3 + A_5t^5 + A_7t^7 + A_9t^9 + \ldots.
\end{aligned}
$$

By substitution of these values in the differential equations and the equating of each resulting coefficient to zero we arrive at the following equations:

$$
\begin{aligned}
&1 . 2A_2 = 2v_0 + 3x_0 - x_0^{-2}, \\
&2 . 3A_3 = - 4A_2 - x_0^{-3}v_0, \\
&3 . 4A_4 = 6A_3 + 3A_2 + \tfrac{1}{2}x_0^{-4}(3v_0^2 + 4x_0A_2), \\
&4 . 5A_5 = - 8A_4 + \tfrac{3}{2}x_0^{-5}v_0(v_0^2 + 2x_0A_2) - x_0^{-3}A_3, \\
&5 . 6A_6 = 10A_5 + 3A_4 + \tfrac{1}{2}x_0^{-4}(6v_0A_3 + 4x_0A_4 + 3A_2^2) - \tfrac{3}{8}x_0^{-6}(v_0^2 + 2x_0A_2)(5v_0^2 + 6x_0A_2), \\
&6 . 7A_7 = - 12A_6 + \tfrac{3}{2}x_0^{-5}v_0(2v_0A_3 + 2x_0A_4 + A_2^2) - \tfrac{15}{8}x_0^{-7}v_0(v_0^2 + 2x_0A_2)^2 \\
&\qquad + \tfrac{3}{2}x_0^{-5}(v_0^2 + 2x_0A_2)A_3 - x_0^{-3}A_5,
\end{aligned}
$$

$$
\begin{aligned}
7.8A_8 = {} & 14A_7 + 3A_6 + \tfrac{1}{2}x_0^{-4}(6v_0A_5 + 4x_0A_6 + 6A_2A_4 + A_3^2) \\
& - \tfrac{3}{4}x_0^{-6}(v_0^2 + 2x_0A_2)(10v_0A_3 + 8x_0A_4 + A_2^2) \\
& + \tfrac{5}{16}x_0^{-8}(v_0^2 + 2x_0A_2)^2(7v_0^2 + 8x_0A_2) \\
& + \tfrac{3}{2}x_0^{-6}(2v_0A_3 + 2x_0A_4 + A_2^2)A_2, \\
8.9A_9 = {} & -16A_8 + \tfrac{3}{2}x_0^{-5}v_0(2v_0A_6 + 2x_0A_6 + 2A_2A_4 + A_3^2) \\
& - \tfrac{15}{4}x_0^{-7}v_0(v_0^2 + 2x_0A_2)(2v_0A_3 + 2x_0A_4 + A_2^2) \\
& + \tfrac{35}{16}x_0^{-9}v_0(v_0^2 + 2x_0A_2)^3 + \tfrac{3}{2}x_0^{-5}(2v_0A_3 + 2x_0A_4 + A_2^2)A_3 \\
& - \tfrac{15}{8}x_0^{-7}(v_0^2 + 2x_0A_2)^2A_3 + \tfrac{3}{2}x_0^{-5}(v_0^2 + 2x_0A_2)A_6 - x_0^{-3}A_7.
\end{aligned}
$$

By means of these relations each A can be derived from all the A which precede it.

When the axis is that of y, the series have the forms

$$
\begin{aligned}
x &= v_0t + A_3t^3 + A_5t^5 + A_7t^7 + A_9t^9 + \dots, \\
y &= y_0 + A_2t^2 + A_4t^4 + A_6t^6 + A_8t^8 + \dots.
\end{aligned}
$$

And the equations, determining the coefficients A, are

$$
\begin{aligned}
1.2A_2 &= -2v_0 - y_0^{-2}, \\
2.3A_3 &= 4A_2 + 3v_0 - y_0^{-3}v_0, \\
3.4A_4 &= -6A_3 + \tfrac{1}{2}y_0^{-4}(3v_0^2 + 4y_0A_2), \\
4.5A_5 &= 8A_4 + 3A_3 + \tfrac{3}{2}y_0^{-5}v_0(v_0^2 + 2y_0A_2) - y_0^{-3}A_3.
\end{aligned}
$$

The equations are not written as far as in the former case, as it is evident they may be derived from the preceding group by putting y_0 in the place of x_0, reversing the signs of the first terms, and removing the term $3A_{n-2}$ from the equations, which give the values of the A of even subscripts, into those which give the values of the A of odd subscripts, after having augmented the subscript by unity.

The velocity of the moon of maximum lunation vanishes in quadratures, and when $v_0 = 0$ the preceding series become, putting $y_0^{-3} = a$,

$$
\begin{aligned}
x = y_0\Big[& -\tfrac{1}{3}at^3 + \left(\tfrac{1}{60}a - \tfrac{1}{60}a^2\right)t^5 + \left(-\tfrac{1}{2520}a + \tfrac{1}{315}a^2 + \tfrac{1}{280}a^3\right)t^7 \\
& + \left(\tfrac{1}{181440}a - \tfrac{1}{12096}a^2 + \tfrac{1}{45360}a^3 + \tfrac{47}{90720}\right)t^9 \\
& + \left(-\tfrac{1}{19958400}a + \tfrac{317}{9979200}a^2 - \tfrac{13}{120960}a^3 + \tfrac{10403}{4989600}a^4 + \tfrac{947}{237600}a^6\right)t^{11}\Big]. \\
y = y_0\Big[& 1 - \tfrac{1}{2}at^2 + \left(\tfrac{1}{6}a - \tfrac{1}{12}a^2\right)t^4 + \left(-\tfrac{1}{180}a + \tfrac{1}{60}a^2 - \tfrac{11}{360}a^3\right)t^6 \\
& + \left(\tfrac{1}{10080}a - \tfrac{1}{1008}a^2 + \tfrac{13}{1120}a^3 - \tfrac{73}{5040}a^4\right)t^8 \\
& + \left(-\tfrac{1}{907200}a + \tfrac{17}{90720}a^2 - \tfrac{1}{756}a^3 + \tfrac{4608}{453600}a^4 - \tfrac{3}{400}a^6\right)t^{10}\Big].
\end{aligned}
$$

These series suffice for computing the values of x and y with the desired exactitude when t is less than 0. 3.

This special case of the moon of maximum lunation will now be treated. As there seems to be no ready method of getting even a roughly approximate

value of y_0, we are reduced to making a series of guesses. I first took $y_0 = 0.82$; tracing the path to its intersection with the axis of x, σ_1, which ought to be 270°, came out 261° 29′ 47″. 9. A second trial was made with $y_0 = 0.7937$; the result was $\sigma_1 = 267° \; 37' \; 8''.3$. Again a third trial with $y_0 = 0.7835$ gave $\sigma_1 = 269° \; 41' \; 13''.3$. The principal data acquired in the three trials are given in the following lines:

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	σ_1.	Maximum Variation.
0.8200	0.972430	—0.339523	—0.288149	—1.927275	261° 29′ 47″.9	44° 57′ 4″
0.7937	0.908207	—0.290945	—0.089184	—2.144832	267 37 8 .3	46 39 36
0.7835	0.884782	—0.274324	—0.012170	—2.227928	269 41 13 .3	47 17 21.

T denotes the time employed in crossing the quadrant, and the last column contains the maximum value of the angular deviation of the body from its mean direction as seen from the origin, that is, the direction it would have had, had it moved across the quadrant with a uniform angular velocity about the origin.

A check may be had on the accuracy of the computations by mechanical quadratures. We determine the value of the constant $2C$ which completes Jacobi's integral from the coordinates and velocities, both at the beginning and at the end of the motion, for each of the three trials. The result is

y_0.	First value.	Second value.
0.8200	2.43902	2.43901
0.7937	2.51985	2.51987
0.7835	2.55265	2.55261

We can now apply Lagrange's general interpolation formula to these data, and, regarding σ_1 as the independent variable, inquire what are the values which correspond to $\sigma_1 = 270°$. The numbers of the first trial must be multiplied by $+ 0.014861$; those of the second by $- 0.210190$; those of the third by $+ 1.195329$, and the sums taken. The results are

y_0.	T.	x_1.	$\frac{dx_1}{dt}$.	$\frac{dy_1}{dt}$.	$2C$.	Maximum Variation.
0.781898	0.881160	0.271798	—0.000083	—2.24093	2.55788	47° 23′ 12″.

That $\frac{dx_1}{dt}$ does not rigorously vanish is due to the employment of only three terms in the interpolation; for the same reason the value of $2C$ does not quite agree with that obtained from the values of x_1 and $\frac{dy_1}{dt}$. To make all these elements accordant we add 0.00009 to the value of $\frac{dy_1}{dt}$.

A table of approximate values of x and y, derived roughly from the data afforded by the process of mechanical quadratures is appended: they will serve for plotting the orbit.

t.	x.	y.	t.	x.	y.	t.	x.	y.
0.00	—.0000	+.7819	0.30	—.0148	+.7080	0.60	—.1177	+.4748
0.02	.0000	.7816	0.32	.0180	.6978	0.62	.1294	.4519
0.04	.0000	.7806	0.34	.0215	.6869	0.64	.1418	.4277
0.06	.0001	.7790	0.36	.0256	6752	0.66	.1547	.4022
0.08	.0003	.7767	0.38	.0301	.6629	0.68	.1680	.3752
0.10	.0005	.7737	0.40	.0351	.6499	0.70	.1818	.3466
0.12	.0009	.7701	0.42	.0407	.6361	0.72	.1956	.3162
0.14	.0015	.7659	0.44	.0468	.6216	0.74	.2095	.2839
0.16	.0022	.7610	0.46	.0534	.6063	0.76	.2230	.2496
0.18	.0032	.7554	0.48	.0607	.5902	0.78	.2359	.2131
0.20	.0044	.7492	0.50	.0686	.5733	0.80	.2475	.1745
0.22	.0058	.7423	0.52	.0771	.5555	0.82	.2575	.1339
0.24	.0076	.7347	0.54	.0863	.5369	0.84	.2653	.0913
0.26	.0096	.7265	0.56	.0961	.5172	0.86	.2704	.0474
0.28	.0120	.7176	0.58	.1066	.4965	0.88	.2718	.0027

The following is the table of the numerical values of the quantities of principal interest belonging to the moons mentioned at the beginning of this paragraph. In the first line stands the earth's moon, having very approximately $12\frac{59}{160}$ lunations in the period of its primary. In the last line is the moon of maximum lunation. The quantities belonging to the moon of two lunations have been somewhat rudely inferred from the numbers in the adjacent lines.

Number of Lunations in period of Primary. $\frac{1}{m}$.	Radius Vector in Syzygies. r_0.	Radius Vector in Quadratures. r_1.	Ratio. $\frac{r_1}{r_0}$.	Velocity in Syzygies. v_0.	Velocity in Quadratures. v_1.	Ratio. $\frac{v_1}{v_0}$.	$2C$.	Maximum Variation.
$12\frac{59}{160}$	0.17610	0.17864	1.01446	2.22295	2.16484	0.97386	6.50888	0° 35′ 6″
10	0.19965	0.20418	1.02271	2.06163	1.97693	0.95892	5.88686	0 55 21
9	0.21209	0.21813	1.02849	1.98730	1.88501	0.94853	5.61562	1 9 33
8	0.22652	0.23485	1.03678	1.90904	1.78250	0.93372	5.33873	1 29 58
7	0.24342	0.25543	1.04934	1.82721	1.66572	0.91162	5.05535	2 0 53
6	0.26332	0.28167	1.06969	1.74333	1.52851	0.87677	4.76409	2 50 49
5	0.28660	0.31699	1.10605	1.66247	1.35953	0.81777	4.46103	4 18 37
4	0.31232	0.36897	1.18138	1.60111	1.13480	0.70876	4.13277	7 17 0
3	0.33235	0.45973	1.38329	1.62141	0.79387	0.48962	3.72018	14 34 14
2	0.302	0.684	2.26	2.00	0.18	0.09	2.89	37 21
1.78265	0.27180	0.78190	2.87676	2.24102	0.00000	0.00000	2.55788	47 23 12

In regard to this table we may notice the following points. The moon of the last line is the most remarkable: it is, of the class of satellites considered in this chapter, (viz., those which have the radius vector at a minimum in syzygies, and at a maximum in quadratures,) that which, having the longest lunation, is still able to appear at all angles with the sun, and thus undergo all possible phases. Whether this class of satellites is properly to be prolonged beyond this moon, can only be decided by further employment of mechanical quadratures. But it is at least certain that the orbits, if they do exist, do not intersect the line of quadratures, and that the moons describing them would make oscillations to and fro, never departing as much as 90° from the point of conjunction or of opposition.

This moon is also remarkable for becoming stationary with respect to the sun when in quadrature; and its angular motion near this point is so nearly equal to that of the sun that, for about one-third of its lunation, it is within 1° of quadrature. From the data of the table we learn that such a moon, circulating about the earth, would make a lunation in 204.896 days.

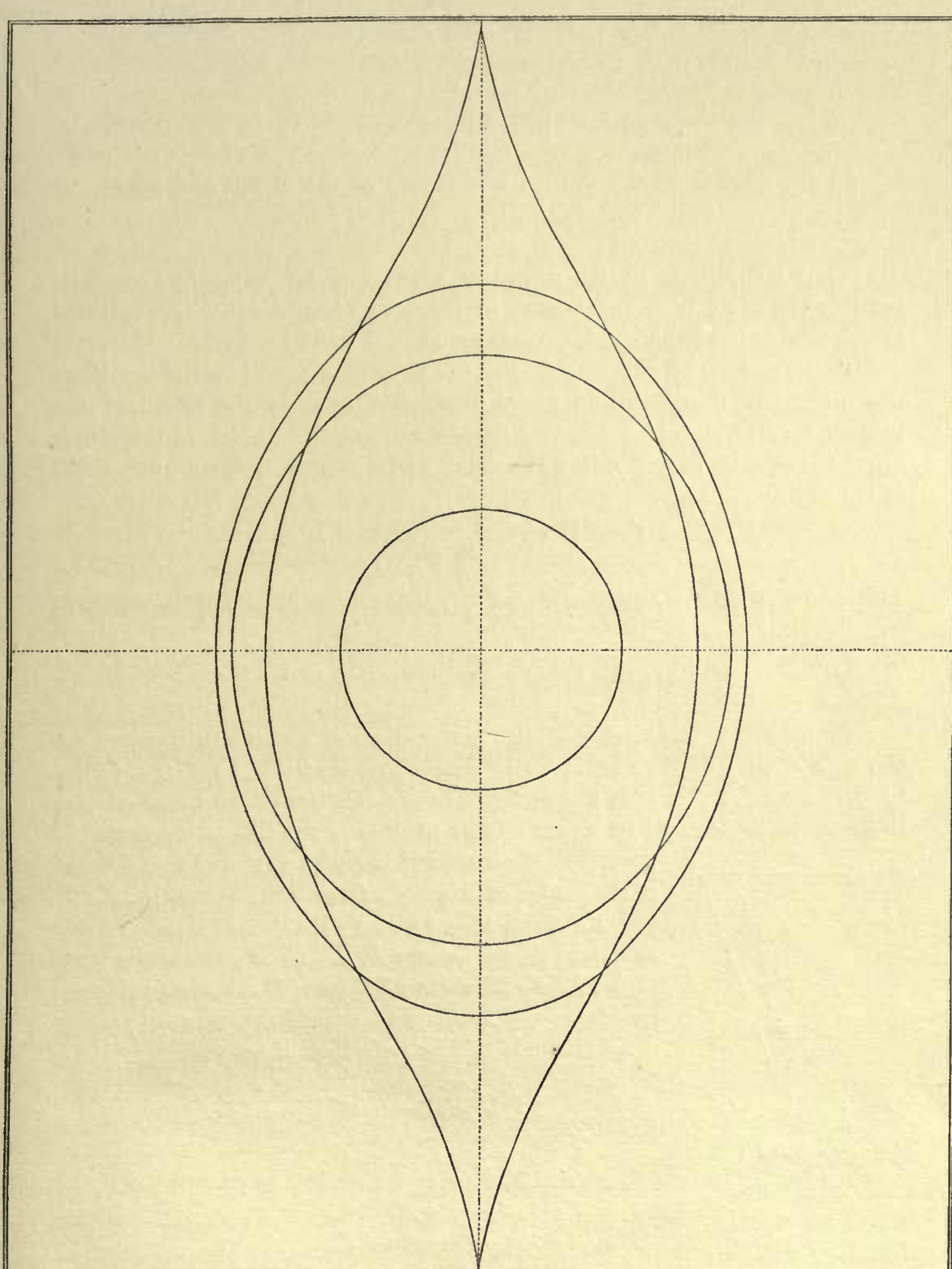
We notice that the radius vector in syzygies of this class of satellites arrives at a maximum before we reach the moon of maximum lunation. This maximum value is very nearly, if not exactly, $\frac{1}{2}$, when measured in terms of our linear unit, and thus is a little less than double the radius vector of the earth's moon. It occurs in the case of the moon which has about 2.8 lunations in the period of its primary.

The radius vector in quadratures augments continuously as the length of the lunation increases, as also does the ratio of these radii, until, in the moon of maximum lunation, the radius in quadratures is but little less than three times that in syzygies.

The velocity in syzygies does not continuously diminish, but attains a minimum somewhere about the moon of four lunations, and afterwards augments so that, for the moon of maximum lunation, it does not differ greatly from the velocity of the earth's moon in syzygies. On the other hand the velocity in quadratures constantly diminishes.

The maximum value of the variation augments rapidly with increase in the length of lunation, so that, in the moon of maximum lunation, it exceeds an octant, or is more than 80 times the value which belongs to the earth's moon.

In the adjoining figure are constructed graphically the paths of the earth's moon, of the moons of four and three lunations, and of the moon of maximum lunation. The moons in the first lines of the table have paths which approach the ellipse quite closely, but the paths of the moons of the last lines exhibit considerable deviation from this curve, while the orbit of the moon of maximum lunation has sharp cusps at the points of quadrature.



MEMOIR No. 33.

On the Motion of the Centre of Gravity of the Earth and Moon.

(Analyst, Vol. V, pp. 33-38, 1878.)

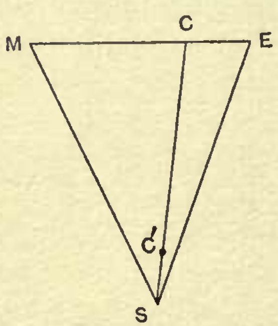
That the motion of the centre of gravity of the earth and moon is sensibly the same as if the masses of these two bodies were concentrated at this centre has been confidently asserted over and over again. However a little scepticism on the matter may not be altogether ill-advised. Were this assertion true it would follow that, setting aside the action of the planets, we should get the sensibly exact mean angular motion of this centre about the sun, by first deriving the mean distance a' from the elliptic value of the radius vector,

$$r' = a'\left[1 + \tfrac{1}{2}e'^2 + \text{periodic terms}\right],$$

and then n' from the equation $n' = \sqrt{\frac{M}{a'^3}}$, M denoting the sum of the masses of the sun, earth and moon.

Let us see whether this value is sensibly exact under the conditions we suppose.

Assume that the masses of the sun, earth and moon are denoted by m_1, m_2 and m_3, and their rectangular coordinates severally by ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; ξ_3, η_3, ζ_3. And let the rectangular coordinates of the moon relative to the earth be denoted by x, y, z; those of the sun relative to the centre of gravity of the earth and moon by x', y', and z'; and those of the centre of gravity of the three bodies by X, Y, and Z. Then from an attentive consideration of the subjoined figure, where S, E and M denote the positions of the sun, earth and moon, C the centre of gravity of the last two bodies, and C' the centre of gravity of all three, it will be seen that, if we put



$$\mu = \frac{m_3}{m_2 + m_3}, \qquad \mu' = \frac{m_2 + m_3}{m_1 + m_2 + m_3},$$

we shall have

$$\xi_1 = \mu' x' + X,$$
$$\xi_2 = (\mu' - 1)\,x' - \mu x + X,$$
$$\xi_3 = (\mu' - 1)\,x' + (1 - \mu)\,x + X,$$

with two groups, of three equations each, for the η and ζ, obtained from these by writing, in the second members, for x and X, y and Y, and again z and Z.

If we differentiate the equations just written, then square and add the results, after having multiplied them severally by m_1, m_2, and m_3, we shall get

$$m_1 d\xi_1^2 + m_2 d\xi_2^2 + m_3 d\xi_3^2 = m_1\mu' dx'^2 + m_2\mu dx^2 + M dX^2.$$

From this equation it is evident that, if Ω denote the potential function, the differential equations, determining the variables x, y, z, x', y', z', are

$$m_2\mu\,\frac{d^2x}{dt^2} = \frac{\partial\Omega}{\partial x}, \qquad m_2\mu\,\frac{d^2y}{dt^2} = \frac{\partial\Omega}{\partial y}, \qquad m_2\mu\,\frac{d^2z}{dt^2} = \frac{\partial\Omega}{\partial z},$$
$$m_1\mu'\,\frac{d^2x'}{dt^2} = \frac{\partial\Omega}{\partial x'}, \qquad m_1\mu'\,\frac{d^2y'}{dt^2} = \frac{\partial\Omega}{\partial y'}, \qquad m_1\mu'\,\frac{d^2z'}{dt^2} = \frac{\partial\Omega}{\partial z'}.$$

Hence it may be gathered that the disturbing function for the motion of the sun relative to the centre of gravity of the earth and moon differs from the corresponding function for the motion of the moon relative to the earth only by a constant factor which depends on the masses.

The expression for Ω is

$$\Omega = \frac{m_1 m_2}{\Delta_{1,2}} + \frac{m_1 m_3}{\Delta_{1,3}} + \frac{m_2 m_3}{\Delta_{2,3}},$$

where the Δ's are given by the equations

$$\Delta_{1,2}^2 = (x' + \mu x)^2 + (y' + \mu y)^2 + (z' + \mu z)^2,$$
$$\Delta_{1,3}^2 = [x' - (1-\mu)x]^2 + [y' - (1-\mu)y]^2 + [z' - (1-\mu)z]^2,$$
$$\Delta_{2,3}^2 = x^2 + y^2 + z^2.$$

Let us put

$$r^2 = x^2 + y^2 + z^2, \qquad r'^2 = x'^2 + y'^2 + z'^2, \qquad rr'S = xx' + yy' + zz'.$$

Then

$$\Delta_{1,2}^2 = r'^2 + 2\mu rr'S + \mu^2 r^2,$$
$$\Delta_{1,3}^2 = r'^2 - 2(1-\mu)\,rr'S + (1-\mu)^2 r^2.$$

Since the ratio $\frac{r}{r'}$ is only about $\frac{1}{400}$, and μ about $\frac{1}{80}$, it is convenient to expand, in Ω, the reciprocals of $\Delta_{1,2}$ and $\Delta_{1,3}$ in infinite series proceeding according to ascending powers of $\frac{r}{r'}$. This, in both cases, evidently depends on the development of

$$(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$$

in powers of α. By the Theorem of Lagrange, in solving the equation $y - \alpha F(y) = x$ with respect to y, we get

$$y = x + \alpha F(x) + \frac{\alpha^2}{1.2}\frac{d\,.\,F(x)^2}{dx} + \ldots + \frac{\alpha^n}{n!}\frac{d^{n-1}.\,F(x)^n}{dx^{n-1}} + \ldots,$$

whence

$$\frac{dy}{dx} = 1 + \alpha\frac{d.\,F(x)}{dx} + \frac{\alpha^2}{1.2}\frac{d^2.\,F(x)^2}{dx^2} + \ldots + \frac{\alpha^n}{n!}\frac{d^n.\,F(x)^n}{dx^n} + \ldots$$

Let us suppose that we have here $F(y) = \frac{1}{2}(y^2 - 1)$; the equation, on which y depends, becomes then $y - \frac{1}{2}\alpha(y^2 - 1) = x$, and the resolution of this quadratic in y gives $1 - \alpha y = \sqrt{1 - 2\alpha x + \alpha^2}$, and, by differentiation, $\frac{dy}{dx} = (1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$. Consequently,

$$\begin{aligned}(1-2\alpha x+\alpha^2)^{-\frac{1}{2}} &= 1 + \frac{\alpha}{2}\frac{d(x^2-1)^2}{dx} + \frac{\alpha^2}{2.4}\frac{d^2(x^2-1)^2}{dx^2} + \ldots + \frac{\alpha^n}{2\,.\,.\,2^n}\frac{d^n(x^2-1)^n}{dx^n} + \ldots \\ &= 1 + \alpha\frac{2}{2}x \\ &\quad + \alpha^2\left[\frac{4.3}{2.4}x^2 - \frac{2}{1}\cdot\frac{2.1}{2.4}\right] \\ &\quad + \alpha^3\left[\frac{6.5.4}{2.4.6}x^3 - \frac{3}{1}\cdot\frac{4.3.2}{2.4.6}x\right] \\ &\quad + \alpha^4\left[\frac{8.7.6.5}{2.4.6.8}x^4 - \frac{4}{1}\cdot\frac{6.5.4.3}{2.4.6.8}x^2 + \frac{4.3}{1.2}\cdot\frac{4.3.2.1}{2.4.6.8}\right] \\ &\quad + \ldots\ldots\ldots\end{aligned}$$

The law of the numerical coefficients in this series is so plain that we can set down as many terms as we have occasion for.

In making the application to the reciprocals of $\Delta_{1,2}$ and $\Delta_{1,3}$ we must put, in the first case, $\alpha = -\mu\frac{r}{r'}$, in the second, $\alpha = (1-\mu)\frac{r}{r'}$, and in both $x = S$.

We obtain as the potential function proper for the relative motion of the moon about the earth,

$$\begin{aligned}\frac{1}{m_2\mu}\Omega &= \frac{m_2 + m_3}{r} + m_1\left[\frac{1}{\mu}\frac{1}{\Delta_{1,2}} + \frac{1}{1-\mu}\frac{1}{\Delta_{1,3}}\right] \\ &= \frac{m_2 + m_3}{r} \\ &\quad + m_1\left\{\left[(1-\mu)^{-1} + \mu^{-1}\right]\frac{1}{r'}\right.\end{aligned}$$

$$+\left[(1-\mu)+\mu\right]\frac{r^2}{r'^3}\left[\frac{4.3}{2.4}S^2-\frac{2}{1}\cdot\frac{2.1}{2.4}\right]$$
$$+\left[(1-\mu)^2-\mu^2\right]\frac{r^3}{r'^4}\left[\frac{6.5.4}{2.4.6}S^3-\frac{3}{1}\cdot\frac{4.3.2}{2.4.6}S\right]$$
$$+\left[(1-\mu)^3+\mu^3\right]\frac{r^4}{r'^5}\left[\frac{8.7.6.5}{2.4.6.8}S^4-\frac{4}{1}\cdot\frac{6.5.4.3}{2.4.6.8}S^2+\frac{4.3}{1.2}\cdot\frac{4.3.2.1}{2.4.6.8}\right]$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big\}.$$

To get the similar function for the relative motion of the sun about the centre of gravity of the earth and moon, it is necessary to multiply the preceding expression by

$$\frac{m_2\mu}{m_1\mu'}=\frac{m_1+m_2+m_3}{m_1}\mu(1-\mu).$$

The term of the potential function for the moon, factored by $\frac{r^3}{r'^4}$, gives rise to inequalities in the lunar coordinates factored by $\frac{a}{a'}$. As this term has $1-2\mu$ as a factor, we see the correctness of the rule which directs to multiply this class of inequalities by $1-2\mu$, in order to include the effect of the disturbance of the relative motion of the sun about the earth by the lunar mass.

In treating the motion of the sun about the centre of gravity of the earth and moon, it will suffice to take two terms of the preceding expression and put

$$\frac{1}{m_1\mu'}\Omega=\frac{M}{r'}+M\mu(1-\mu)\frac{r^2}{r'^3}\left(\tfrac{3}{2}S^2-\tfrac{1}{2}\right).$$

Let the longitudes of the sun and moon be denoted respectively by λ' and λ, and neglect the latitudes; then

$$\frac{1}{m_1\mu'}\Omega=\frac{M}{r'}+\tfrac{1}{4}M\mu(1-\mu)\frac{r^2}{r'^3}[3\cos 2(\lambda-\lambda')+1].$$

The differential equations, determining r' and λ', are

$$\frac{d^2r'}{dt^2}-r'\frac{d\lambda'^2}{dt^2}+\frac{M}{r'^2}+\tfrac{3}{4}n'^2a'\mu(1-\mu)\frac{a^2}{a'^2}[3\cos 2\tau+1]=0,$$
$$\frac{d}{dt}\left(r'^2\frac{d\lambda'}{dt}\right)-\tfrac{3}{2}n'^2a'^2\mu(1-\mu)\frac{a^2}{a'^2}\sin 2\tau=0,$$

where, it will be noticed, we have put $r=a$, and, after differentiation, in the final small terms, $r'=a'$, $\frac{M}{a'^3}=n'^2$, and $\lambda-\lambda'=\tau$ the mean angular distance of the moon from the sun. The integration of the second equation gives

$$\frac{d\lambda'}{dt} = \frac{a_0^2 n'}{r'^2} - \tfrac{3}{4}\frac{n'^2}{n-n'}\mu(1-\mu)\frac{a^2}{a'^2}\cos 2\tau,$$

a_0 being the arbitrary constant. We can now eliminate $\frac{d\lambda'}{dt}$ from the first equation, and we get

$$\frac{d^2r'}{dt^2} + \frac{M}{r'^2} - \frac{a_0^4 n'^2}{r'^3} + \tfrac{3}{4}n'^2 a'\mu(1-\mu)\frac{a^2}{a'^2}\left[\frac{3n-n'}{n-n'}\cos 2\tau + 1\right] = 0.$$

Let us suppose that this equation is satisfied by

$$r' = a_0 + a'a_1\cos 2\tau,$$

a_1 being a coefficient to be determined. Substituting this value of r' in the differential equation, we get the two equations of condition,

$$\frac{M}{a_0^2} - n'^2 a_0 + \tfrac{3}{4}n'^2 a'\mu(1-\mu)\frac{a^2}{a'^2} = 0,$$

$$(4n^2 - 8nn' + 3n'^2)\,a_1 - \tfrac{3}{4}n'^2\frac{3n-n'}{n-n'}\mu(1-\mu)\frac{a^2}{a'^2} = 0.$$

Whence may be derived

$$\frac{a_0}{a'} = 1 + \tfrac{1}{4}\mu(1-\mu)\frac{a^2}{a'^2},$$

$$a_1 = \tfrac{3}{4}\frac{m^2(3-m)}{(1-m)(4-8m+3m^2)}\mu(1-\mu)\frac{a^2}{a'^2},$$

where, as is usually done in the lunar theory, we have put $\frac{n'}{n} = m$. The value of r', thus obtained, being substituted in the expression for $\frac{d\lambda'}{dt}$, we get

$$\frac{d\lambda'}{dt} = n' - \tfrac{3}{4}n'\frac{m}{1-m}\,\frac{4-2m+m^2}{4-8m+3m^2}\mu(1-\mu)\frac{a^2}{a'^2}\cos 2\tau.$$

Integrating

$$\lambda' = \varepsilon' + n't - \tfrac{3}{8}\frac{m^2}{(1-m)^2}\,\frac{4-2m+m^2}{4-8m+3m^2}\mu(1-\mu)\frac{a^2}{a'^2}\sin 2\tau.$$

The numerical values of the constant quantities, which enter into these formulas, are

$$m = 0.0748, \qquad \mu = \frac{1}{82.4869}, \qquad \frac{a}{a'} = 0.002587, \qquad n' = 1295977''.4.$$

They give us

$$r' = a'\,[1.00000\,00200 + 0.00000\,00003\cos 2\tau],$$

$$\lambda' = \varepsilon' + n't - 0''.0001\sin 2\tau.$$

The periodic terms of these equations are too small for consideration, but the constant term of $\frac{r'}{a'}$ may be noticed. If we should obtain the value of a' from measured values of r' on the assumption that the value of the constant term is unity, it would be too large by the 0.00000 002 part. And this value substituted in the equation $n' = \sqrt{\frac{M}{a'^3}}$ would give n' too small by the 0.00000 003 part, or n' would be too small by $0''.03895$; or the error in the mean longitude of the sun would amount to nearly $4''$ in a century, a quantity which could not, in the present state of astronomy, be neglected. However, it is only fair to state that astronomers proceed in a way the reverse of this; that is, they observe n' and thence deduce a', and in this case the term 0.00000 002 is without significance, since the logarithms of the radii vectores in the ephemerides are usually given to 7 decimals only.

MEMOIR No. 34.

The Secular Acceleration of the Moon.

(The Analyst, Vol. V, pp. 105–110, 1878.)

In the Philosophical Transactions for 1853, Prof. J. C. Adams, of Cambridge University, England, showed that the values of the secular acceleration of the mean motion of the moon, obtained by Plana and Damoisean, were erroneous, for the simple reason that these authors had, inadvertently, made the solar eccentricity constant throughout a certain portion of the investigation. This statement of Prof. Adams gave rise to an animated and prolonged controversy, the history of which will, no doubt, always possess much interest.

It is proposed to obtain here the coefficient of the term in the moon's mean motion involving the square of the solar eccentricity, supposed variable, to quantities of the order of the square of the sun's disturbing force, when the lunar eccentricity and inclination of orbit are neglected. The method employed has no novelty, having been used before by Mr. Donkin. But, at the end of the investigation, I have found that it is possible to do without an explicit development of R in a periodic series, and thus the treatment is, to a considerable degree, abbreviated.

Let ζ denote the mean longitude of the moon as affected by this secular inequality, and n_0 the mean motion at a given epoch taken as the origin of time; we propose to prove that, in the equation

$$\frac{d\zeta}{dt} = n = n_0\left[1 + H(e'^2_0 - e'^2)\right],$$

the true value of H is

$$\tfrac{3}{2}\left(\frac{n'}{n_0}\right)^2 - \frac{3771}{64}\left(\frac{n'}{n_0}\right)^4.$$

Employing the method of variation of the elements, we have, for determining the four elements n, ζ, e and ω of the lunar orbit, these equations

$$\frac{dn}{dt} = -\frac{3}{\mu a^2}\frac{\partial R}{\partial \zeta}, \qquad \frac{de}{dt} = -\frac{na}{\mu e}\frac{\partial R}{\partial \omega},$$

$$\frac{d\zeta}{dt} = n - 2\frac{na^2}{\mu}\frac{\partial R}{\partial a} + \tfrac{1}{2}\frac{nae}{\mu}\frac{\partial R}{\partial e} \qquad \frac{d\omega}{dt} = \frac{na}{\mu e}\frac{\partial R}{\partial e}.$$

In writing them, all terms, multiplied by higher powers of e than the first, have been neglected, as they are not needed in obtaining H to the

degree of accuracy proposed. It may be noted that R is taken with such a sign that $\frac{\partial R}{\partial x}$ denotes the force tending to increase x.

Since we need not retain any terms multiplied by the ratio $\frac{a}{a'}$, the value of R is

$$R = \tfrac{1}{4} n'^2 r^2 \frac{a'^3}{r'^3} [1 + 3 \cos 2 (\lambda - \lambda')],$$

where λ and λ' are the true longitudes of the moon and sun. The constant part of R is evidently the same as that of $\frac{1}{4} n'^2 a^2 \frac{a'^3}{r'^3}$, when we reject e^2, that is, it is equal to $\frac{1}{4} n'^2 a^2 (1 + \frac{3}{2} e'^2)$.

Considering first those terms in R which are independent of e (we need those multiplied by e only when taking account of the effects produced by the variations δe and $\delta \omega$), we see that the only terms in R which produce terms in $\frac{dn}{dt}$, and, consequently, can give rise to terms independent of sines or cosines of arguments in $\frac{d\zeta}{dt}$, have arguments of the form $2\zeta + \psi$, where ψ denotes an angle depending on the sun's mean motion. Hence, denoting any one of these terms of R by $n'^2 a^2 A \cos (2\zeta + \psi)$, where A is independent of the lunar elements, but will generally contain e'^2, and regard being had to this term alone, the equations determining the elements become

$$\frac{dn}{dt} = 6 n'^2 A \sin (2\zeta + \psi), \qquad \frac{d\zeta}{dt} = n - 4 \frac{n'^2}{n} A \cos (2\zeta + \psi),$$

where μ has been eliminated by using the equation $\mu = n^2 a^3$. Integrating these, and considering ψ as constant, since its variability affects only the terms in R multiplied by $\frac{n'^5}{n_0^5}$, but regard being had to the variability of a through e', where we may consider $\frac{d.e'^2}{dt}$ as constant, we obtain

$$\delta n = -3 \frac{n'^2}{n} A \cos (2\zeta + \psi) + \frac{3}{2} \frac{n'^2}{n^2} \frac{dA}{d.e'^2} \frac{d.e'^2}{dt} \sin (2\zeta + \psi),$$

$$\delta \zeta = -\frac{7}{2} \frac{n'^2}{n^2} A \sin (2\zeta + \psi) - \frac{5}{2} \frac{n'^2}{n^3} \frac{dA}{d.e'^2} \frac{d.e'^2}{dt} \cos (2\zeta + \psi).$$

This being only a first approximation in which we have had regard only to quantities of the order of n'^2, we proceed to a second approximation. And first, in the expression for $\frac{dn}{dt}$, we substitute for ζ, $\zeta + \delta\zeta$; and we find, for

that part of the increment which is independent of the sines or cosines of arguments, the expression

$$\delta . \frac{dn}{dt} = -15 \frac{n'^4}{n^3} A \frac{dA}{d.e'^2} \frac{d.e'^2}{dt}.$$

Integrating this and putting $e'^2 - e'^2_0 = \delta . e'^2$,

$$\delta n = -\tfrac{15}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta . e'^2.$$

Again, in the expression for $\frac{d\zeta}{dt}$, increasing n and ζ by their variations δn and $\delta\zeta$,

$$\delta . \frac{d\zeta}{dt} = -\tfrac{15}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta . e'^2 - 6 \frac{n'^4}{n^3} A^2 - 14 \frac{n'^4}{n^3} A^2.$$

Now the constant part of this value of $\delta . \frac{d\zeta}{dt}$ goes to form part of the constant n_0, hence, desiring to retain only the varying part, we may write $\frac{d.A^2}{d.e'^2} \delta . e'^2$ for A^2, and thus obtain

$$\delta . \frac{d\zeta}{dt} = -\tfrac{55}{2} \frac{n'^4}{n^3} \frac{d.A^2}{d.e'^2} \delta . e'^2. \qquad (1)$$

In the next place let us consider the terms in R multiplied by e; they are all of the form

$$n'^2 a^2 e A \cos(\omega + k\zeta + \psi),$$

where A and ψ possess the same quality as before, and k may be -3, -1 or 1. Representing $\omega + k\zeta + \psi$ by θ, the equations determining the elements are, regard being had to this term alone,

$$\frac{dn}{dt} = 3kn'^2 Ae \sin\theta, \qquad \frac{de}{dt} = \frac{n'^2}{n} A \sin\theta,$$

$$\frac{d\zeta}{dt} = n - \tfrac{3}{2} \frac{n'^2}{n} Ae \cos\theta, \qquad \frac{d\theta}{dt} = \frac{n'^2}{n} A \frac{1}{e} \cos\theta.$$

In the last equation we have written only the term divided by e, since this alone can produce terms $\delta . \frac{d\zeta}{dt}$ of the kind we seek. Integrating the last two as we integrated in the former case, we obtain

$$\delta e = -\frac{1}{k} \frac{n'^2}{n^2} A \cos\theta + \frac{1}{k^2} \frac{n'^2}{n^3} \frac{d.A^2}{d.e'^2} \frac{d.e'^2}{dt} \sin\theta,$$

$$\delta\theta = \frac{1}{ke} \frac{n'^2}{n^2} A \sin\theta + \frac{1}{k^2 e} \frac{n'^2}{n^3} \frac{d.A^2}{d.e'^2} \frac{d.e'^2}{dt} \cos\theta.$$

Augmenting, in the expression for $\frac{dn}{dt}$, e and θ by these quantities, we obtain, regard being had only to the terms which are independent of sines or cosines of angles,

$$\delta \cdot \frac{dn}{dt} = \frac{3}{2k} \frac{n'^4}{n^3} \frac{d \cdot A^2}{d \cdot e'^2} \frac{d \cdot e'^2}{dt}.$$

Increasing, in the expression for $\frac{d\zeta}{dt}$, the elements n, e and θ by their variations δn, δe and $\delta\theta$, and preserving only the terms independent of the sines or cosines of angles, we get

$$\delta \cdot \frac{d\zeta}{dt} = \frac{3}{2k} \frac{n'^4}{n^3} \frac{d \cdot A^2}{d \cdot e'^2} \delta \cdot e'^2 + \frac{7}{2k} \frac{n'^4}{n^3} A^2.$$

In like manner as before, rejecting the constant part of this which coalesces with n_0, we obtain

$$\delta \cdot \frac{d\zeta}{dt} = \frac{5}{k} \frac{n'^4}{n^3} \frac{d \cdot A^2}{d \cdot e'^2} \delta \cdot e'^2. \tag{2}$$

When formulas (1) and (2) are applied to all the terms of R, to which each is applicable, and the results added, we shall have the complete value of $\delta \cdot \frac{d\zeta}{dt}$, since it is plain that the combination of two different terms in R will always produce terms in $\delta \cdot \frac{d\zeta}{dt}$ involving the sines or cosines of angles.

Denoting the mean anomalies of the moon and sun by ξ and ξ', and the mean angular distance of the bodies by τ, the part of $a\,R$, which is independent of e, may be written

$$aR = A_0 + A_1 \cos 2\tau + A_2 \cos \xi' + A_3 \cos (2\tau - \xi') + A_4 \cos (2\tau + \xi').$$

Formula (1) applied to this series gives

$$\delta \cdot \frac{d\zeta}{dt} = -\tfrac{55}{2} n \frac{d\,(A_1^2 + A_3^2 + A_4^2)}{d \cdot e'^2} \delta \cdot e'^2.$$

We can obtain the terms in R multiplied by e from the series just given by using the equation

$$\begin{aligned} \frac{\partial R}{\partial e} &= r \frac{\partial R}{\partial r} \frac{d \cdot \log r}{de} + \frac{\partial R}{\partial \lambda} \frac{d\lambda}{de} \\ &= -2\, R \cos \xi + 2 \frac{\partial R}{\partial \tau} \sin \xi. \end{aligned}$$

Whence

$$\frac{\partial . aR}{\partial e} = -2A_0 \cos \xi - 3A_1 \cos (2\tau - \xi) + A_1 \cos (2\tau + \xi)$$
$$- A_2 \cos (\xi - \xi') - A_2 \cos (\xi + \xi')$$
$$-3A_3 \cos (2\tau - \xi' - \xi) + A_3 \cos (2\tau - \xi' + \xi)$$
$$-3A_4 \cos (2\tau + \xi' - \xi) + A_4 \cos (2\tau + \xi' + \xi).$$

Applying formula (2) to this series gives

$$\delta . \frac{d\zeta}{dt} = n\frac{d}{d . e'^2}[-5\,(4A_0^2 + A_2^2 + A_2^2) - \tfrac{5}{8}(A_1^2 + A_3^2 + A_4^2) + 5\,(9A_1^2 + 9A_3^2 + 9A_4^2)]\,\delta . e'^2$$
$$= n\frac{d}{d . e'^2}[\tfrac{130}{3}(A_1^2 + A_3^2 + A_4^2) - 10\,(2A_0^2 + A_2^2)]\,\delta . e'^2.$$

Adding this to the expression given by formula (1),

$$\delta . \frac{d\zeta}{dt} = n\frac{d}{d . e'^2}[\tfrac{95}{6}\,(A_1^2 + A_2^2 + A_4^2) - 10\,(2A_0^2 + A_2^2)]\,\delta . e'^2.$$

But, denoting the constant term of a^2R^2 by K, we have

$$K = A_0^2 + \tfrac{1}{2}(A_1^2 + A_2^2 + A_3^2 + A_4^2).$$

Or

$$A_1^2 + A_3^2 + A_4^2 = 2K - (2A_0^2 + A_2^2),$$

and

$$\delta . \frac{d\zeta}{dt} = n\frac{d}{d . e'^2}[\tfrac{95}{3}K - \tfrac{155}{6}(2A_0^2 + A_2^2)]\,\delta . e'^2.$$

But we have

$$a^2R^2 = \tfrac{1}{16}\frac{n'^4}{n^4}\frac{a'^6}{r'^6}[\tfrac{11}{2} + 6\cos 2(\lambda - \lambda') + \tfrac{9}{2}\cos 4(\lambda - \lambda')],$$

and hence K is equal to the constant term of $\tfrac{11}{32}\dfrac{n'^4}{n^4}\dfrac{a'^6}{r'^6}$. In consequence, denoting the constant term of $\dfrac{a'^6}{r'^6}$ by L, we shall have

$$\delta . \frac{d\zeta}{dt} = n\frac{a}{d . e'^2}\left[\tfrac{1045}{96}\frac{n'^4}{n^4}L - \tfrac{155}{6}(2A_0^2 + A_2^2)\right]\delta . e'^2.$$

Also we evidently have

$$\tfrac{1}{4}\frac{n'^2}{n^2}\frac{a'^3}{r'^3} = A_0 + A_2 \cos \xi',$$

and thus

$$\tfrac{1}{8}\frac{n'^4}{n^4}L = 2A_0^2 + A_2^2.$$

Substituting this value

$$\delta . \frac{d\zeta}{dt} = n\frac{d}{d . e'^2}\left[\tfrac{245}{32}\frac{n'^4}{n^4}L\right]\delta . e'^2.$$

But the constant term of $\frac{a'^6}{r'^6}$ is known to be $1 + \frac{15}{2}e'^2$; hence, in fine,

$$\delta . \frac{d\zeta}{dt} = \frac{3675}{64} \frac{n'^4}{n^3} \delta . e'^2 .$$

To obtain $\frac{d\zeta}{dt}$ we must add to n both this term and that which arises, in the first approximation, from the term $-\frac{2na^2}{\mu} \frac{\partial R}{\partial a}$ in the differential equation for $\frac{d\zeta}{dt}$, which is therefore equal to the constant term of $-\frac{n'^2}{n} \frac{a'^3}{r'^3}$, that is, to $-\frac{n'^2}{n}(1 + \frac{3}{2}e'^2)$. Thus

$$\frac{d\zeta}{dt} = n - \frac{n'^2}{n}(1 + \tfrac{3}{2}e'^2_0) - \left(\frac{3}{2}\frac{n'^2}{n} - \frac{3675}{64}\frac{n'^4}{n^3}\right)\delta . e'^2 .$$

We could have added to the first two terms of this equation a term $B \frac{n'^4}{n^3} e'^2_0$, where B is a numerical coefficient, equal to the aggregate of the constants we have virtually neglected whenever we wrote $\delta . e'^2$ for e'^2, but it will be easily seen that this would not change the final result. We evidently have

$$n_0 = n - \frac{n'^2}{n}(1 + \tfrac{3}{2}e'^2_0) .$$

From which, to a sufficient degree of approximation,

$$n = n_0 + \frac{n'^2}{n_0} .$$

Substituting this value of n, we get

$$\frac{d\zeta}{dt} = n_0 \left[1 + \left(\frac{3}{2}\frac{n'^2}{n_0^2} - \frac{3771}{64}\frac{n'^4}{n_0^4}\right)(e'^2_0 - e'^2)\right] .$$

MEMOIR No. 35.

Note on Hansen's General Formulae for Perturbations.

(American Journal of Mathematics, Vol. IV, pp. 256–259, 1881.)

The last form in which HANSEN expressed the perturbations of the mean anomaly and equated radius vector is exhibited by the following equations:

$$n_0 z = n_0 t + c_0 + \int \left\{ \overline{W} + \frac{h_0}{h}\left(\frac{\nu}{1+\nu}\right)^2 \right\} n_0 dt,$$

$$\nu = C - \tfrac{1}{2}\int \overline{\left(\frac{dW}{d\zeta}\right)} dt,$$

(Equations 36 and 37, p. 97.)*

It will be perceived that the right-hand member of the first of these involves three quantities, viz. $\overline{W}$, ν and $\frac{h_0}{h}$. But the last of these quantities has no share in defining the position of the body, and it is desirable to get rid of it, provided that can be done without complicating the equation. This is readily accomplished by means of the equation (33, p. 95)

$$\frac{dz}{dt} = \frac{h_0}{h(1+\nu)^2}.$$

The result is

$$n_0 z = n_0 t + c_0 + \int \frac{\overline{W} + \nu^2}{1-\nu^2} n_0 dt. \qquad = n_0 t + c_0 + \int [\overline{W}(1+\nu^2) + \nu^2]$$

Why HANSEN has not put the equation in this form I cannot imagine; the advantage, not only as regards simplicity of expression, but also in point of ease of computation, is evident.

HANSEN develops $\overline{W}$ by TAYLOR'S theorem, and, limiting ourselves to the second power of the disturbing force, we have

$$\overline{W} = \overline{W}_0 + \overline{\left(\frac{dW_0}{d\gamma}\right)} n_0 \delta z = \overline{W}_0 - 2\frac{d\nu}{dt}\delta z.$$

When this value is substituted for $\overline{W}$ in the equation for $n_0 z$, we have a differential equation of the first order and degree for the determination of δz,

*See *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten. Von P. A. Hansen. Erste Abhandlung.* (*Abhandlungen der Königlichen Sächsischen Gesellschaft der Wissenschaften. Band III.*) The numbering of the equations and the paging are from this volume.

the integral of which is well known. Terms of three dimensions with respect to disturbing forces being neglected, this procedure furnishes the equation

$$n_0 \delta z = (1 - 2\nu) \int [(1 + 2\nu) \overline{W}_0 + \nu^2]\, n_0 dt,$$

which, however, is without interest other than analytical, as its use involves more labor than that of the equation given by HANSEN.

HANSEN'S equation for the determination of ν has the disadvantage of not affording the constant term of this quantity, and is inconvenient in computing the portion, of the form

$$At + Bt^2 + Ct^3 + \ldots,$$

which is independent of the arguments g, g', &c., as the values of A, B, &c., must be determined to a degree of accuracy much beyond what is necessary in the case of the other terms. As all the arbitrary constants admissible have been introduced by the integrations which give z, it is evident there must exist an equation determining ν without additional integrations. HANSEN has virtually employed this in the place where he shows how the constant term of ν is to be obtained, but has nowhere given it explicitly. This lacuna I propose to fill here.

The equation 39, p. 97,

$$W_0 = 2\frac{h}{h_0} - \frac{h_0}{h} - 1 + 2\frac{h}{h_0}\xi\frac{\rho}{a_0}\cos\omega + 2\frac{h}{h_0}\eta\frac{\rho}{a_0}\sin\omega,$$

may be employed to discover the value of $\frac{h}{h_0}$. The known expressions for $\frac{\rho}{a_0}\cos\omega$ and $\frac{\rho}{a_0}\sin\omega$ are

$$\frac{\rho}{a_0}\cos\omega = -\tfrac{3}{2}e + \left(J_{\frac{e}{2}}^{(0)} - J_{\frac{e}{2}}^{(2)}\right)\cos\gamma + \tfrac{1}{2}\left(J_e^{(1)} - J_e^{(3)}\right)\cos 2\gamma + \ldots,$$

$$\frac{\rho}{a_0}\sin\omega = \left(J_{\frac{e}{2}}^{(0)} + J_{\frac{e}{2}}^{(2)}\right)\sin\gamma + \tfrac{1}{2}\left(J_e^{(1)} + J_e^{(3)}\right)\sin 2\gamma + \ldots,$$

where HANSEN'S notation for the BESSELIAN function is employed, and the subscript zero, which properly belongs to e, is, for convenience in writing, omitted. In his memoirs, where the mean anomaly is employed as the independent variable, HANSEN directs to compute only the parts of W_0 which are independent of γ or which have $\pm\gamma$ in their arguments; that is, the parts which have the form

$$X_0 + X_1\cos\gamma + X_2\sin\gamma,$$

X_0, X_1 and X_2 being independent of γ.

It will be easily perceived that, if we put

$$P = \tfrac{3}{2} \frac{e}{J_{\frac{e}{2}}^{(0)} - J_{\frac{e}{2}}^{(2)}},$$

P being thus a constant, the three first terms of W_0 must have the value

$$2\frac{h}{h_0} - \frac{h_0}{h} - 1 = X_0 + PX_1.$$

In this equation we may substitute for $\frac{h_0}{h}$ its value obtained from equation 33, and thus we obtain

$$\frac{2}{(1+\nu)^2\left(1+\frac{d.\delta z}{dt}\right)} - (1+\nu)^2\left(1+\frac{d.\delta z}{dt}\right) - 1 = X_0 + PX_1.$$

This equation, when $\frac{d.\delta z}{dt}$ is known, gives ν without additional integrations. To put it into a form suitable for computation, we add to each member such a quantity as will make the first equal to -6ν, then dividing both members by -6 we get

$$\nu = -\tfrac{1}{6}[X_0 + PX_1] - \tfrac{1}{2}\left[(1+\nu)^2\frac{d.\delta z}{dt} + \nu^2\right] + \frac{\left[(1+\nu)^2\frac{d.\delta z}{dt} + 2\nu + \nu^2\right]^2}{3(1+\nu)^2\left(1+\frac{d.\delta z}{dt}\right)}$$

This equation is rigorous. If we may restrict ourselves to terms of the first order with respect to disturbing forces, it reduces to

$$\nu = -\tfrac{1}{6}[X_0 + PX_1] - \tfrac{1}{2}\frac{d.\delta z}{dt},$$

or, if terms of the second order must be included, to

$$\nu = -\tfrac{1}{6}[X_0 + PX_1] - \tfrac{1}{2}\frac{d.\delta z}{dt} + \tfrac{1}{3}\left[\frac{d.\delta z}{dt} + \tfrac{1}{2}\nu\right]^2 + \tfrac{3}{4}\nu^2.$$

The function usually tabulated is com. log $(1+\nu)$; and we have

$$\text{com. log}\,(1+\nu) = M\left\{-\tfrac{1}{6}[X_0 + PX_1] - \tfrac{1}{2}\frac{d.\delta z}{dt} + \tfrac{1}{3}\left[\frac{d.\delta z}{dt} + \tfrac{1}{2}\nu\right]^2 + \tfrac{1}{4}\nu^2\right\},$$

M being the modulus of common logarithms.

These equations are as readily used as those given by HANSEN, and are free from the disadvantages, previously mentioned, which belong to the latter. All the quantities involved, except X_1, have already been obtained in the computation of δz. Also X_1 is readily got by putting $\gamma = 0$ in the terms of W_0 which involve this quantity, and summing two and two together the terms which result.

MEMOIR No. 36.

Notes on the Theories of Jupiter and Saturn.

(The Analyst, Vol. VIII, pp. 33–40, 89–93, 1881.)

On account of their large masses and the near approach to commensurability of their mean motions, Jupiter and Saturn offer the most interesting, as well as the most difficult, field for research in the planetary perturbations of the solar system. In the following remarks, without treating the subject in a complete manner, which would be impossible here, I intend only to point out a method of procedure and give a few illustrations of its use.

At present we shall notice only the introaction of the sun, Jupiter and Saturn. It will facilitate matters much if we employ differential equations in which the potential function is the same for both planets. This is accomplished by an orthogonal transformation of variables. Let us suppose that the coordinates of the sun in space are denoted by

$$X, Y \text{ and } Z,$$

those of Jupiter by

$$X + x + \varkappa x', \quad Y + y + \varkappa y' \text{ and } Z + z + \varkappa z',$$

and those of Saturn by

$$X + x' + \varkappa x, \quad Y + y' + \varkappa y \text{ and } Z + z' + \varkappa z,$$

where $\varkappa$ is a small constant to be so determined that the variables $x, x', \ldots$ may be orthogonal.

M, *m* and *m′* denoting severally the masses of the sun, Jupiter and Saturn, the *vis viva* *T* of the system is represented by the equation

$$2Tdt^2 = MdX^2 + m\,(dX + dx + \varkappa dx')^2 + m'\,(dX + dx' + \varkappa dx)^2$$
$$+ \text{ similar terms in } Y, y, y' \text{ and } Z, z, z'$$
$$= (M + m + m')\left(dX + \frac{m + \varkappa m'}{M + m + m'}\,dx + \frac{m' + \varkappa m}{M + m + m'}\,dx'\right)^2$$
$$+ \left(m + \varkappa^2 m' - \frac{(m + \varkappa m')^2}{M + m + m'}\right)dx^2$$
$$+ \left(m' + \varkappa^2 m - \frac{(m' + \varkappa m)^2}{M + m + m'}\right)dx'^2$$
$$+ 2\left(\varkappa\,(m + m') - \frac{(m + \varkappa m')\,(m' + \varkappa m)}{M + m + m'}\right)dx\,dx'$$
$$+ \text{ similar terms in } Y, y, y', Z, z, z'.$$

In order that the system of variables may be orthogonal, the coefficient of $dxdx'$ in this expression must vanish, which gives us, for the determination of $\varkappa$, the quadratic equation

$$\varkappa^2 - \left(\frac{M}{m} + \frac{M}{m'} + 2\right)\varkappa + 1 = 0 .$$

Of this the smaller root must be taken. Employing Bessel's values of the masses of Jupiter and Saturn, $\frac{M}{m} = 1047.879$, $\frac{M}{m'} = 3501.6$. Hence

$$\varkappa^2 - 4551.479\,\varkappa + 1 = 0,$$

whence we get

$$\varkappa = \frac{1}{4551.479} + \left(\frac{1}{4551.479}\right)^3 + \ldots = 0.0002197088 .$$

For brevity we will put

$$\mu = m + \varkappa^2 m' - \frac{(m + \varkappa m')^2}{M + m + m'},$$

$$\mu' = m' + \varkappa^2 m - \frac{(m' + \varkappa m)^2}{M + m + m'}.$$

When the numerical values are substituted these equations give

$$\mu = 0.9990467623\, m, \qquad \mu' = 0.9997145123\, m' .$$

As we do not wish to know X, Y and Z, but only the six variables x, y, z, x', y' and z', which assign the positions of Jupiter and Saturn relatively to the sun, we can altogether neglect the first term in the last expression for T, and write

$$T = \mu\, \frac{dx^2 + dy^2 + dz^2}{2dt^2} + \mu'\, \frac{dx'^2 + dy'^2 + dz'^2}{2dt^2}.$$

If we put

$$x^2 + y^2 + z^2 = r^2, \qquad x'^2 + y'^2 + z'^2 = r'^2, \qquad xx' + yy' + zz' = rr's,$$

the expression of the potential function is

$$\Omega = \frac{Mm}{[r^2 + 2\varkappa rr's + \varkappa^2 r'^2]^{\frac{1}{2}}} + \frac{Mm'}{[r'^2 + 2\varkappa rr's + \varkappa^2 r^2]^{\frac{1}{2}}} + \frac{mm'}{1 - \varkappa}\, \frac{1}{[r'^2 - 2rr's + r^2]^{\frac{1}{2}}}.$$

From the last term it will be seen that the motion of two planets, whose coordinates are severally x, y, z and x', y', z', relatively to each other, is homothetic with the relative motion of Jupiter and Saturn.

The differential equations of motion are

$$\frac{d^2x}{dt^2}=\frac{1}{\mu}\frac{\partial\Omega}{\partial x}, \qquad \frac{d^2y}{dt^2}=\frac{1}{\mu}\frac{\partial\Omega}{\partial y}, \qquad \frac{d^2z}{dt^2}=\frac{1}{\mu}\frac{\partial\Omega}{\partial z},$$

$$\frac{d^2x'}{dt^2}=\frac{1}{\mu'}\frac{\partial\Omega}{\partial x'}, \qquad \frac{d^2y'}{dt^2}=\frac{1}{\mu'}\frac{\partial\Omega}{\partial y'}, \qquad \frac{d^2z'}{dt^2}=\frac{1}{\mu'}\frac{\partial\Omega}{\partial z'}.$$

The first two radicals in Ω may be expanded in series proceeding according to ascending powers of $\varkappa$; and, since this constant is so small, the cube and higher powers of it may be neglected. Thus

$$\Omega=\frac{Mm}{r}+\frac{Mm'}{r'}-\varkappa Mm\frac{r'}{r^2}s-\varkappa Mm'\frac{r}{r'^2}s-\tfrac{1}{2}\varkappa^2Mm\frac{r'^2}{r^3}(1-3s^2)$$
$$-\tfrac{1}{2}\varkappa^2Mm'\frac{r^2}{r'^3}(1-3s^2)+\frac{mm'}{1-\varkappa}\frac{1}{[r'^2-2rr's+r^2]^{\frac{1}{2}}}.$$

If for Ω are substituted only the first two terms of this expression, the differential equations are easily integrated, and the variables x, y, z and x', y', z' represent the motion of two planets moving according to the laws of elliptic motion, whose mean motions are

$$\sqrt{\frac{Mm}{\mu a^3}} \text{ and } \sqrt{\frac{Mm'}{\mu' a'^3}}.$$

In terms of symbols whose meaning is well known, we will put

$$L=\sqrt{[Mm\mu a]}, \qquad L'=\sqrt{[Mm'\mu' a']},$$
$$G=\sqrt{[Mm\mu a(1-e^2)]}, \qquad G'=\sqrt{[Mm'\mu' a'(1-e'^2)]},$$
$$H=\sqrt{[Mm\mu a(1-e^2)]}\cos i, \qquad H'=\sqrt{[Mm'\mu' a'(1-e'^2)]}\cos i',$$

and denote the mean anomalies by l and l', the distances of the perihelia from the nodes by g and g' and the longitudes of the nodes by h and h', and moreover, put

$$R=\frac{Mm}{2a}+\frac{Mm'}{2a'}-\varkappa M\left(m\frac{r'}{r^2}+m'\frac{r}{r'^2}\right)s-\tfrac{1}{2}\varkappa^2M\left(m\frac{r'^2}{r^3}+m'\frac{r^2}{r'^3}\right)(1-3s^2)$$
$$+\frac{mm'}{1-\varkappa}\frac{1}{[r'^2-2rr's+r^2]^{\frac{1}{2}}}.$$

We have then the following system of differential equations for determining the elements L, G, H, L', G', H', l, g, h, l', g', h':—

$$\frac{dL}{dt}=\frac{\partial R}{\partial l}, \qquad \frac{dl}{dt}=-\frac{\partial R}{\partial L}, \qquad \frac{dL'}{dt}=\frac{\partial R}{\partial l'}, \qquad \frac{dl'}{dt}=-\frac{\partial R}{\partial L'},$$
$$\frac{dG}{dt}=\frac{\partial R}{\partial g}, \qquad \frac{dg}{dt}=-\frac{\partial R}{\partial G}, \qquad \frac{dG'}{dt}=\frac{\partial R}{\partial g'}, \qquad \frac{dg'}{dt}=-\frac{\partial R}{\partial G'},$$
$$\frac{dH}{dt}=\frac{\partial R}{\partial h}, \qquad \frac{dh}{dt}=-\frac{\partial R}{\partial H}, \qquad \frac{dH'}{dt}=\frac{\partial R}{\partial h'}, \qquad \frac{dh'}{dt}=-\frac{\partial R}{\partial H'},$$

in which it is understood that R is expressed in terms of these elements.

As r is a function of the three elements L, G, l only, and r' of L', G', l' only, it follows that the six elements H, g, h, H', g' and h' enter in R only through s; hence we have the equations

$$\frac{\partial R}{\partial g} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial g}, \qquad \frac{\partial R}{\partial H} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial H}, \qquad \frac{\partial R}{\partial h} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial h},$$
$$\frac{\partial R}{\partial g'} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial g'}, \qquad \frac{\partial R}{\partial H'} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial H'}, \qquad \frac{\partial R}{\partial h'} = \frac{\partial R}{\partial s}\frac{\partial s}{\partial h'}.$$

The expression for s being given by

$$rr's = xx' + yy' + zz',$$

and v and v' denoting the true anomalies, the rectangular coordinates have the equivalents

$$\begin{aligned}
x &= r[\cos h \cos (v + g) - \cos i \sin h \sin (v + g)], \\
y &= r[\sin h \cos (v + g) + \cos i \cos h \sin (v + g)], \\
z &= r \sin i \sin (v + g), \\
x' &= r'[\cos h' \cos (v' + g') - \cos i' \sin h' \sin (v' + g')], \\
y' &= r'[\sin h' \cos (v' + g') + \cos i' \cos h' \sin (v' + g')], \\
z' &= r' \sin i' \sin (v' + g').
\end{aligned}$$

Whence the following expression for s,

$$\begin{aligned}
s = \cos (h - h') \cos (v + g) \cos (v' + g') &+ \cos i \cos i' \cos (h - h') \sin (v + g) \sin (v' + g') \\
&+ \cos i \sin (h - h') \cos (v + g) \sin (v' + g') \\
&- \cos i \sin (h - h') \sin (v + g) \cos (v' + g') \\
&+ \sin i \sin i' \sin (v + g) \sin (v' + g').
\end{aligned}$$

Remembering that v and v' contain only the same elements as r and r', and that

$$\cos i = \frac{H}{G}, \qquad \sin i = \frac{\sqrt{G^2 - H^2}}{G}, \qquad \cos i' = \frac{H'}{G'}, \qquad \sin i' = \frac{\sqrt{G'^2 - H'^2}}{G'},$$

it will be found that

$$\begin{aligned}
&\frac{d}{dt}[\sqrt{G^2 - H^2} \cos h + \sqrt{G'^2 - H'^2} \cos h'] = 0, \\
&\frac{d}{dt}[\sqrt{G^2 - H^2} \sin h + \sqrt{G'^2 - H'^2} \sin h'] = 0, \\
&\frac{d}{dt}[H + H'] = 0.
\end{aligned}$$

Hence we have the following integrals of the differential equations,

$$\begin{aligned}
\sqrt{G^2 - H^2} \cos h + \sqrt{G'^2 - H'^2} \cos h' &= \text{a constant}, \\
\sqrt{G^2 - H^2} \sin h + \sqrt{G'^2 - H'^2} \sin h' &= \text{a constant}, \\
H + H' &= \text{a constant}.
\end{aligned}$$

These integrals may be employed to diminish the number of differential equations. Thus far the system of planes to which x, y, z . . . are referred has been left indeterminate; let us now assume that the plane of maximum areas, called by Laplace the invariable plane, is chosen for the plane of xy. In this case it is well known that the constants of the first two of the integrals, given above, become zero. Then we shall have

$$\sqrt{G^2 - H^2}\cos h + \sqrt{G'^2 - H'^2}\cos h' = 0,$$
$$\sqrt{G^2 - H^2}\sin h + \sqrt{G'^2 - H'^2}\sin h' = 0,$$
$$H + H' = c,$$

c being an arbitrary constant. But, since i and i' are supposed contained between 0° and 180°, the radicals in these expressions must be taken positively. Consequently the equations are equivalent to

$$h' = h + 180°, \qquad H + H' = c, \qquad H - H' = \frac{G^2 - G'^2}{c}.$$

These equations determine the values of the elements H, H' and h' in terms of the rest, and they may be used to eliminate them from R. Then it is plain, from the expression of s, given above, that h will also disappear from R, and we shall have

$$R = \text{function } (L, G, L', G', l, g, l', g'),$$

and s takes the much simpler form

$$s = -\cos(v - v' + g - g') + \frac{(G + G')^2 - c^2}{2GG'}\sin(v + g)\sin(v' + g').$$

As to the partial derivatives of R with respect to L, L', l, l', g, g', they are evidently unchanged by this elimination of the elements H, H', h, h'. But $\left(\frac{\partial R}{\partial G}\right)$ and $\left(\frac{\partial R}{\partial G'}\right)$ denoting the derivatives of R on the supposition of its containing the elements H, H', h, h', we have

$$\left(\frac{\partial R}{\partial G}\right) = \frac{\partial R}{\partial G} - \frac{\partial R}{\partial H}\frac{\partial H}{\partial G} - \frac{\partial R}{\partial H'}\frac{\partial H'}{\partial G},$$
$$\left(\frac{\partial R}{\partial G'}\right) = \frac{\partial R}{\partial G'} - \frac{\partial R}{\partial H}\frac{\partial H}{\partial G'} - \frac{\partial R}{\partial H'}\frac{\partial H'}{\partial G'}.$$

But we also have

$$\frac{\partial R}{\partial H} - \frac{\partial R}{\partial H'} = \frac{d(h' - h)}{dt} = 0,$$

hence

$$\left(\frac{\partial R}{\partial G}\right) = \frac{\partial R}{\partial G} - \frac{\partial R}{\partial H}\frac{\partial(H + H')}{\partial G} = \frac{\partial R}{\partial G},$$
$$\left(\frac{\partial R}{\partial G'}\right) = \frac{\partial R}{\partial G'} - \frac{\partial R}{\partial H}\frac{\partial(H + H')}{\partial G'} = \frac{\partial R}{\partial G'}.$$

Moreover

$$\frac{\partial R}{\partial c}=\frac{\partial R}{\partial H}\frac{\partial H}{\partial c}+\frac{\partial R}{\partial H'}\frac{\partial H'}{\partial c}=\frac{\partial R}{\partial H}\frac{\partial(H+H')}{\partial c}=\frac{\partial R}{\partial H}.$$

Thus the system of differential equations still retains its canonical form, and is

$$\frac{dL}{dt}=\frac{\partial R}{\partial l},\quad \frac{dL'}{dt}=\frac{\partial R}{\partial l'},\quad \frac{dG}{dt}=\frac{\partial R}{\partial g},\quad \frac{dG'}{dt}=\frac{\partial R}{\partial g'},$$

$$\frac{dl}{dt}=-\frac{\partial R}{\partial L},\quad \frac{dl'}{dt}=-\frac{\partial R}{\partial L'},\quad \frac{dg}{dt}=-\frac{\partial R}{\partial G},\quad \frac{dg'}{dt}=-\frac{\partial R}{\partial G'}.$$

After this system of eight differential equations is integrated, the value of h is found by a quadrature from the equation

$$\frac{dh}{dt}=-\frac{\partial R}{\partial c}.$$

These integrations introduce nine arbitrary constants which, together with c, make ten. The reference of the coordinates to any arbitrary planes introduces three more, but one of these coalesces with the constant which completes the value of h.

The time t does not explicitly enter R, hence the complete derivative of it with respect to t is

$$\frac{dR}{dt}=\frac{\partial R}{\partial L}\frac{dL}{dt}+\frac{\partial R}{\partial l}\frac{dl}{dt}+\ldots$$

If, in this are substituted the values of $\frac{dL}{dt}$, $\frac{dl}{dt}$, . . . , from the equations just given, we shall find that it vanishes; hence

$$R = \text{a constant}$$

is an integral of the system of differential equations. This integral may be employed to eliminate one of the elements, as L, from the equations. We can also take one of the elements, as l, for the independent variable in place of t. The system of equations, to be integrated, is then reduced to the ix

$$\frac{dL'}{dl}=-\frac{\frac{\partial R}{\partial l'}}{\frac{\partial R}{\partial L}},\qquad \frac{dG}{dl}=-\frac{\frac{\partial R}{\partial g}}{\frac{\partial R}{\partial L}},\qquad \frac{dG'}{dl}=-\frac{\frac{\partial R}{\partial g'}}{\frac{\partial R}{\partial L}},$$

$$\frac{dl'}{dl}=\frac{\frac{\partial R}{\partial L'}}{\frac{\partial R}{\partial L}},\qquad \frac{dg}{dl}=\frac{\frac{\partial R}{\partial G}}{\frac{\partial R}{\partial L}},\qquad \frac{dg'}{dl}=\frac{\frac{\partial R}{\partial G'}}{\frac{\partial R}{\partial L}}.$$

A simpler form can be given to them. If the solution of $R =$ a constant gives

$$L = \text{function } (L', G, G', l', g, g', l),$$

and L is supposed to stand for the right member of this, the foregoing equations can be written

$$\frac{dL'}{dl} = \frac{\partial L}{\partial l'}, \qquad \frac{dG}{dl} = \frac{\partial L}{\partial g}, \qquad \frac{dG'}{dl} = \frac{\partial L}{\partial g'},$$
$$\frac{dl'}{dl} = -\frac{\partial L}{\partial L'}, \qquad \frac{dg}{dl} = -\frac{\partial L}{\partial G}, \qquad \frac{dg'}{dl} = -\frac{\partial L}{\partial G'}.$$

When the values of L', G, G', l', g and g' in terms of l have been derived from the integrals of these, they can be substituted in the equation $\frac{dl}{dt} = -\frac{\partial R}{\partial L}$, which will then give t in terms of l, by a quadrature. By inverting this we shall have l in terms of t; and by substituting this in equations previously obtained we shall have the values of all the other elements in terms of t.

It will be noticed that R is a homogeneous function of L, L', G, G' and c of the dimensions -2; hence we shall have

$$L\frac{\partial R}{\partial L} + L'\frac{\partial R}{\partial L'} + G\frac{\partial R}{\partial G} + G'\frac{\partial R}{\partial G'} + c\frac{\partial R}{\partial c} = -2R = \text{a constant},$$

and, as a consequence of this,

$$L\frac{dl}{dt} + L'\frac{dl'}{dt} + G\frac{dg}{dt} + G'\frac{dg'}{dt} + c\frac{dh}{dt} = 2R = \text{a constant}.$$

Thus, if the rate of motion of each angular element l, l' . . ., be multiplied by the linear element which is conjugate to it, the sum of the products is invariable.

The sines of half the inclinations of the orbits on the plane of maximum areas are

$$\sin\frac{i}{2} = \sqrt{\left[\frac{(G + G' - c)(G' - G + c)}{4cG}\right]},$$
$$\sin\frac{i'}{2} = \sqrt{\left[\frac{(G' + G - c)(G - G' + c)}{4cG'}\right]}.$$

Thus, in the special case where the two planets move in the same plane, we have

$$G + G' = c.$$

This equation may be employed to eliminate one of the elements G or G' from R. In the same case, the expression for s is reduced to

$$s = -\cos(v - v' + g - g').$$

Then, if we put

$$G - G' = \Gamma \qquad g - g' = \gamma ,$$

R will be a function of $L, L', \Gamma, l, l', \gamma$, and we shall have, for determining these variables, the system of differential equations

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \qquad \frac{dL'}{dt} = \frac{\partial R}{\partial l'}, \qquad \frac{d\Gamma}{dt} = \frac{\partial R}{\partial \gamma},$$
$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}, \qquad \frac{dl'}{dt} = -\frac{\partial R}{\partial L'}, \qquad \frac{d\gamma}{dt} = -\frac{\partial R}{\partial \Gamma}.$$

After these are integrated, the value of $g + g'$ will be got by a quadrature from the equation

$$\frac{d(g+g')}{dt} = -\frac{\partial R}{\partial c}.$$

If the value of L is obtained from the solution of $R =$ a constant, and we have

$$L = \text{function}\,(L', \Gamma, l', \gamma, l),$$

and l is adopted as the independent variable in place of t, the solution of this special case is reduced to the integration of the four equations

$$\frac{dL'}{dl} = \frac{\partial L}{\partial l'}, \qquad \frac{dl'}{dl} = -\frac{\partial L}{\partial L'}, \qquad \frac{d\Gamma}{dl} = \frac{\partial L}{\partial \gamma}, \qquad \frac{d\gamma}{dl} = -\frac{\partial L}{\partial \Gamma}.$$

The angle between the planes of the orbits of Jupiter and Saturn is about $1\frac{1}{4}°$. This is small enough to make the terms, which are multiplied by the square of the sine of half of it, and which are besides of two or more dimensions with respect to disturbing forces, practically insignificant. Thus, while we are engaged in developing those terms of the coordinates which demand the highest degree of approximation relatively to disturbing forces, we shall assume that the planes coincide; the determination of the effect of non-coincidence of these planes being reserved to the end, when it will be always sufficient to limit ourselves to the first power of the disturbing force.

The coordinates usually preferred by astronomers are the logarithm of the radius vector, the longitude and the latitude. We suppose that the two last are referred to the plane of maximum areas. Let these coordinates be denoted by the symbols $\log . \rho$, λ and β; and let the subscript $({}_0)$ be applied to λ and β when we wish to designate the similar coordinates corresponding to the variables x, y, z, x', y', z'. Then we have

$$\begin{aligned}
\rho \cos\beta \cos\lambda &= r \cos\beta_0 \cos\lambda_0 + \varkappa r' \cos\beta'_0 \cos\lambda'_0 , \\
\rho \cos\beta \sin\lambda &= r \cos\beta_0 \sin\lambda_0 + \varkappa r' \cos\beta'_0 \sin\lambda'_0 , \\
\rho \sin\beta &= r \sin\beta_0 + \varkappa r' \sin\beta'_0 .
\end{aligned}$$

From the first two equations are readily obtained the following two: —

$$\rho \cos\beta \cos(\lambda - \lambda_0) = r \cos\beta_0 + \kappa r' \cos\beta'_0 \cos(\lambda'_0 - \lambda_0),$$
$$\rho \cos\beta \sin(\lambda - \lambda_0) = \kappa r' \cos\beta'_0 \sin(\lambda'_0 - \lambda_0).$$

In the developments in infinite series which follow, the eccentricities of the orbits will be regarded as small quantities of the first order, the squares of the inclinations of the orbits on the plane of maximum areas as quantities of the third order, and κ also as a quantity of the same order. Then all terms, whose order is higher than the sixth, will be neglected. This degree of approximation will be found amply sufficient for the most refined investigations.

Under these conditions, we get

$$\log\rho = \log r + \tfrac{1}{2}\log\left[1 + 2\kappa\frac{r'}{r}s + \kappa^2\frac{r'^2}{r^2}\right].$$
$$= \log r + \kappa\frac{r'}{r}s + \tfrac{1}{2}\kappa^2\frac{r'^2}{r^2}(1 + 2s^2),$$
$$\lambda = \lambda_0 + \kappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda'_0 - \lambda_0) - \tfrac{1}{2}\kappa^2\frac{r'^2}{r^2}\sin 2(\lambda'_0 - \lambda_0),$$
$$\beta = \beta_0 + \kappa\frac{r'}{r}\beta'_0 - \kappa\frac{r'}{r}s\beta_0.$$

We will write η for $\sin\frac{1}{2}i$. Then, to the sufficient degree of approximation,

$$\kappa\frac{r'}{r}s = -\kappa\frac{r'}{r}\cos(v - v' + g - g') + 2\kappa(\eta + \eta')^2\frac{a'}{a}\sin(l + g)\sin(l' + g').$$

In like manner

$$\kappa\frac{r'\cos\beta'_0}{r\cos\beta_0}\sin(\lambda' - \lambda_0) = \kappa(1 + \eta^2 - \eta'^2)\frac{r'}{r}\sin(v - v' + g - g')$$
$$- \kappa\eta^2\frac{a'}{a}\sin(3l - l' + 3g - g') + \kappa\eta'^2\frac{a'}{a}\sin(l + l' + g + g').$$

The expressions for λ_0 and β_0 in terms of elliptic elements are given by Delaunay.* Log r, as well as the following expressions

$$\frac{r'}{a'}\substack{\cos\\ \sin}(v' + g') = -\tfrac{3}{2}e'\substack{\cos\\ \sin}g' + (1 - \tfrac{1}{2}e'^2)\substack{\cos\\ \sin}(l' + g') + (\tfrac{1}{2}e' - \tfrac{3}{8}e'^3)\substack{\cos\\ \sin}(2l' + g')$$
$$+ \tfrac{3}{8}e'^2\substack{\cos\\ \sin}(3l' + g') + \tfrac{1}{3}e'^3\substack{\cos\\ \sin}(4l' + g')$$
$$\pm \tfrac{1}{8}e'^2\substack{\cos\\ \sin}(l' - g') \pm \tfrac{1}{24}e'^3\substack{\cos\\ \sin}(2l' - g'),$$
$$\frac{a}{r}\substack{\cos\\ \sin}(v + g) = -(\tfrac{1}{2}e + \tfrac{1}{8}e^3)\substack{\cos\\ \sin}g + (1 - e^2)\substack{\cos\\ \sin}(l + g) + (\tfrac{3}{2}e - \tfrac{7}{4}e^3)\substack{\cos\\ \sin}(2l + g)$$

* *Théorie du Mouvement de la Lune.* Tom. I, pp. 56–59.

$$+ \tfrac{17}{8} e^2 \begin{matrix}\cos \\ \sin\end{matrix} (3l + g) + \tfrac{71}{24} e^3 \begin{matrix}\cos \\ \sin\end{matrix} (4l + g)$$

$$\mp \tfrac{1}{8} e^2 \begin{matrix}\cos \\ \sin\end{matrix} (l - g) \mp \tfrac{1}{12} e^3 \begin{matrix}\cos \\ \sin\end{matrix} (2l - g),$$

are found in a memoir by Prof. Cayley.* With these data we get

$$\begin{aligned}
\log \rho = \log a &+ \tfrac{1}{4} e^2 + \tfrac{1}{32} e^4 + \tfrac{1}{96} e^6 + x^2 \frac{a'^2}{a^2} \\
&- (e - \tfrac{3}{8} e^3 - \tfrac{1}{64} e^5) \cos l - (\tfrac{3}{4} e^2 - \tfrac{11}{24} e^4 + \tfrac{3}{64} e^6) \cos 2l \\
&- (\tfrac{17}{24} e^3 - \tfrac{77}{128} e^5) \cos 3l - (\tfrac{71}{96} e^4 - \tfrac{129}{160} e^6) \cos 4l \\
&- \tfrac{523}{640} e^5 \cos 5l - \tfrac{899}{960} e^6 \cos 6l \\
&- x \frac{a'}{a} \Big\{ [1 - e^2 - \tfrac{1}{2} e'^2 - (\eta + \eta')^2] \cos (l - l' + g - g') \\
&+ (\tfrac{3}{2} e - \tfrac{7}{4} e^3 - \tfrac{3}{4} ee'^2) \cos (2l - l' + g - g') + (- \tfrac{3}{2} e' + \tfrac{3}{2} e^2 e') \cos (l + g - g') \\
&+ (- \tfrac{1}{2} e - \tfrac{1}{8} e^3 + \tfrac{1}{4} ee'^2) \cos (l' - g + g') + (\tfrac{1}{2} e' - \tfrac{3}{8} e'^3 - \tfrac{1}{2} e^2 e') \cos (l - 2l' + g - g') \\
&+ \tfrac{17}{8} e^2 \cos (3l - l' + g - g') - \tfrac{1}{8} e^2 \cos (l + l' - g + g') \\
&+ \tfrac{3}{4} ee' \cos (g - g') - \tfrac{9}{4} ee' \cos (2l + g - g') \\
&- \tfrac{1}{4} ee' \cos (2l' - g + g') + \tfrac{3}{4} ee' \cos (2l - 2l' + g - g') \\
&+ \tfrac{3}{8} e'^2 \cos (l - 3l' + g - g') + \tfrac{1}{8} e'^2 \cos (l + l' + g - g') \\
&+ \tfrac{71}{24} e^3 \cos (4l - l' + g - g') - \tfrac{1}{12} e^3 \cos (2l + l' - g + g') \\
&- \tfrac{51}{16} e^2 e' \cos (3l + g - g') + \tfrac{3}{16} e^2 e' \cos (l - g + g') \\
&+ \tfrac{17}{16} e^2 e' \cos (3l - 2l' + g - g') - \tfrac{1}{16} e^2 e' \cos (l + 2l' - g + g') \\
&- \tfrac{3}{16} ee'^2 \cos (3l' - g + g') + \tfrac{9}{16} ee'^2 \cos (2l - 3l' + g - g') \\
&- \tfrac{1}{16} ee'^2 \cos (l' + g - g') + \tfrac{3}{16} ee'^2 \cos (2l + l' + g - g') \\
&+ \tfrac{1}{8} e'^3 \cos (l - 4l' + g - g') + \tfrac{1}{24} e'^3 \cos (l + 2l' + g - g') \\
&+ (\eta + \eta')^2 \cos (l + l' + g + g') \Big\} + \tfrac{1}{2} x^2 \frac{a'^2}{a^2} \cos (2l - 2l' + 2g - 2g'),
\end{aligned}$$

$$\begin{aligned}
\lambda = l + g + h &+ (2e - \tfrac{1}{4} e^3 + \tfrac{5}{96} e^5) \sin l + (\tfrac{5}{4} e^2 - \tfrac{11}{24} e^4 + \tfrac{17}{192} e^6) \sin 2l \\
&+ (\tfrac{13}{12} e^3 - \tfrac{43}{64} e^5) \sin 3l + (\tfrac{103}{96} e^4 - \tfrac{451}{480} e^6) \sin 4l \\
&+ \tfrac{1097}{960} e^5 \sin 5l + \tfrac{1223}{960} e^6 \sin 6l \\
&+ (- \eta^2 - \eta^4 + 4\eta^2 e^2) \sin (2l + 2g) + \tfrac{1}{4} \eta^4 \sin (4l + 4g) \\
&+ (- 2\eta^2 e + \tfrac{27}{4} \eta^2 e^3) \sin (3l + 2g) + (2\eta^2 e - \tfrac{7}{4} \eta^2 e^3) \sin (l + 2g) \\
&- \tfrac{13}{4} \eta^2 e^2 \sin (4l + 2g) - \tfrac{3}{4} \eta^2 e^2 \sin 2g \\
&- \tfrac{59}{12} \eta^2 e^3 \sin (5l + 2g) + \tfrac{1}{12} \eta^2 e^3 \sin (l - 2g) \\
&+ x \frac{a'}{a} \Big\{ (1 - e^2 - \tfrac{1}{2} e'^2 + \eta^2 - \eta'^2) \sin (l - l' + g - g') \\
&+ (\tfrac{3}{2} e - \tfrac{7}{4} e^3 - \tfrac{3}{4} ee'^2) \sin (2l - l' + g - g') \\
&+ (\tfrac{1}{2} e + \tfrac{1}{8} e^3 - \tfrac{1}{4} ee'^2) \sin (l' - g + g') + (- \tfrac{3}{2} e' + \tfrac{3}{2} e^2 e') \sin (l + g - g')
\end{aligned}$$

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* Tables of the Development of Functions in the Theory of Elliptic Motion. Mem. Roy. Astr. Soc., Vol. XXIX, p. 191.

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\begin{aligned}
&+ (\tfrac{1}{2} e' - \tfrac{3}{8} e'^2 - \tfrac{1}{2} e^2 e') \sin (l - 2l' + g - g') + \tfrac{17}{8} e^2 \sin (3l - l' + g - g') \\
&+ \tfrac{1}{8} e^2 \sin (l + l' - g + g') + \tfrac{3}{4} ee' \sin (g - g') - \tfrac{9}{4} ee' \sin (2i + g - g') \\
&+ \tfrac{1}{4} ee' \sin (2l' - g + g') + \tfrac{3}{4} ee' \sin (2l - 2l' + g - g') \\
&+ \tfrac{8}{8} e'^2 \sin (l - 3l' + g - g') + \tfrac{1}{8} e'^2 \sin (l + l' + g - g') \\
&+ \tfrac{71}{24} e^3 \sin (4l - l' + g - g') + \tfrac{1}{12} e^3 \sin (2l + l' - g + g') \\
&- \tfrac{51}{16} e^2 e' \sin (3l + g - g') - \tfrac{3}{16} e^2 e' \sin (l - g + g') \\
&+ \tfrac{17}{16} e^2 c' \sin (3l - 2l' + g - g') + \tfrac{1}{16} c^2 e' \sin (l + 2l' - g + g') \\
&+ \tfrac{3}{16} ce'^2 \sin (3l' - g + g') + \tfrac{9}{16} ee'^2 \sin (2l - 3l' + g - g') \\
&- \tfrac{1}{16} ce'^2 \sin (l' + g - g') + \tfrac{3}{16} ee'^2 \sin (2l + l' + g - g') \\
&+ \tfrac{1}{3} e'^3 \sin (l - 4l' + g - g') + \tfrac{1}{24} e'^3 \sin (l + 2l' + g - g') \\
&- \eta^2 \sin (3l - l' + 3g - g') + \eta'^2 \sin (l + l' + g + g') \Big\} \\
&+ \tfrac{1}{2} \varkappa^2 \frac{a'^2}{a^2} \sin (2l - 2l' + 2g - 2g'),
\end{aligned}
$$

$$
\begin{aligned}
\beta = {} & (2\eta - 2\eta e^2 + \tfrac{7}{32} \eta e^4) \sin (l + g) - \tfrac{1}{8} \eta^3 \sin (3l + 3g) + (2\eta e - \tfrac{5}{2} \eta e^3) \sin (2l + g) \\
& - 2\eta e \sin g + (\tfrac{9}{4} \eta e^2 - \tfrac{27}{8} \eta e^4) \sin (3l + g) + (\tfrac{1}{4} \eta e^2 - \tfrac{1}{24} \eta e^4) \sin (l - g) \\
& + \tfrac{8}{3} \eta e^3 \sin (4l + g) + \tfrac{1}{8} \eta e^3 \sin (2l - g) + \tfrac{625}{192} \eta e^4 \sin (5l + g) \\
& + \tfrac{9}{64} \eta e^4 \sin (3l - g) - \eta^3 e \sin (4l + 3g) + \eta^3 e \sin (2l + 3g) \\
& + \varkappa \frac{a'}{a} \Big\{ \eta \sin (2l - l' + 2g - g') + (\eta + 2\eta') \sin (l' + g') \\
& \quad + \tfrac{5}{4} \eta e \sin (3l - l' + 2g - g') - \tfrac{3}{2} \eta e \sin (l - l' + 2g - g') \\
& \quad - \tfrac{3}{2} \eta e' \sin (2l + 2g - g') + \tfrac{1}{2} \eta e' \sin (2l - 2l' + 2g - g') \\
& \quad + \tfrac{1}{2} (\eta + 2\eta') e \sin (l + l' + g') - \tfrac{1}{2} (\eta + 2\eta') e \sin (l - l' - g') \\
& \quad + \tfrac{1}{2} (\eta + 2\eta') e' \sin (2l' + g') - \tfrac{3}{2} (\eta + 2\eta') e' \sin g' \Big\} .
\end{aligned}
$$

As written, these expressions give the coordinates of Jupiter. Those of Saturn are obtained by removing the accent from all the accented symbols, and applying it to those which are unaccented, $\varkappa$ excepted, for which we have $\varkappa' = \varkappa$. Also it is to be remembered that we have $h' = h + 180°$.

The coordinates of the two planets are obtained by employing in these formulas, for the quantities involved in them, the values they actually have at the time in question. The latter are determined by the differential equations previously given; but, instead of integrating these equations in one step, we may, as Delaunay has done in the lunar theory, divide the process into a series of transformations of the variables involved; each of which must be made not only in the expressions for $\log \rho$, λ, β, $\log \rho'$, λ', β', but also in R.

As the introduction of l as the independent variable does not appear to be advantageous, we will suppose that the six variables L, L', Γ, l, l', γ are employed and that t is the independent variable.

Delaunay's method, somewhat amplified, amounts to this: — selecting the argument $\theta = il + i'l' + i''\gamma$, suppose, for the moment, that R is limited to the terms

$$-B - A_1 \cos(il + i'l' + i''\gamma) - A_2 \cos 2(il + i'l' + i''\gamma) + \ldots,$$

where $B, A_1 \ldots$, are functions of L, L' and Γ only. Then if it is found that the differential equations, corresponding to this limited R, are satisfied by the infinite series

$$\begin{aligned}
\theta &= \theta_0(t+c) + \theta_1 \sin[\theta_0(t+c)] + \theta_2 \sin 2[\theta_0(t+c)] + \ldots,\\
l &= (l) + l_0(t+c) + l_1 \sin[\theta_0(t+c)] + l_2 \sin 2[\theta_0(t+c)] + \ldots,\\
l' &= (l') + l'_0(t+c) + l'_1 \sin[\theta_0(t+c)] + l'_2 \sin 2[\theta_0(t+c)] + \ldots,\\
\gamma &= (\gamma) + \gamma_0(t+c) + \gamma_1 \sin[\theta_0(t+c)] + \gamma_2 \sin 2[\theta_0(t+c)] + \ldots,\\
L &= L_0 + L_1 \cos[\theta_0(t+c)] + L_2 \cos 2[\theta_0(t+c)] + \ldots,\\
L' &= L'_0 + L'_1 \cos[\theta_0(t+c)] + L'_2 \cos 2[\theta_0(t+c)] + \ldots,\\
\Gamma &= \Gamma_0 + \Gamma_1 \cos[\theta_0(t+c)] + \Gamma_2 \cos 2[\theta_0(t+c)] + \ldots,
\end{aligned}$$

where c, (l), (l') and (γ) are arbitrary constants, the last three being equivalent to two independent constants, as we have the relation

$$i(l) + i'(l') + i''(\gamma) = 0,$$

and all the other coefficients are known functions of three other constants, a, a' and e, we can replace

$$\begin{aligned}
&L \text{ by } L_0 + L_1 \cos(il + i'l' + i''\gamma) + L_2 \cos 2(il + i'l' + i''\gamma) + \ldots,\\
&L' \text{ by } L'_0 + L'_1 \cos(il + i'l' + i''\gamma) + L'_2 \cos 2(il + i'l' + i''\gamma) + \ldots,\\
&\Gamma \text{ by } \Gamma_0 + \Gamma_1 \cos(il + i'l' + i''\gamma) + \Gamma_2 \cos 2(il + i'l' + i''\gamma) + \ldots,\\
&l \text{ by } l + l_1 \sin(il + i'l' + i''\gamma) + l_2 \sin 2(il + i'l' + i''\gamma) + \ldots,\\
&l' \text{ by } l' + l'_1 \sin(il + i'l' + i''\gamma) + l_2 \sin 2(il + i'l' + i''\gamma) + \ldots,\\
&\gamma \text{ by } \gamma + \gamma_1 \sin(il + i'l' + i''\gamma) + \gamma_2 \sin 2(il + i'l' + i''\gamma) + \ldots,
\end{aligned}$$

and will have, for determining the new variables, l, l', γ, a, a', e, precisely the same differential equations as we started with, provided we make all these substitutions in the function R, and regard the new variables L, L', Γ as connected with a, a', e by the relations

$$\begin{aligned}
L &= L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + \ldots),\\
L' &= L'_0 + \tfrac{1}{2}(\theta_1 L'_1 + 2\theta_2 L'_2 + \ldots),\\
\Gamma &= \Gamma_0 + \tfrac{1}{2}(\theta_1 \Gamma_1 + 2\theta_2 \Gamma_2 + \ldots).
\end{aligned}$$

It will be perceived that, as long as we are dealing with terms of R, whose arguments involve l or l' or both, the second members of the three equations, last written, have values which differ from the elliptic values of L,

L' and Γ only by quantities of the second order with respect to disturbing forces. Hence, if we propose to neglect third order terms, until we have reduced R to a function of the argument γ only, we can assume that L, L' and Γ which are the elements conjugate to the arguments l, l' and γ, are expressed throughout in terms of a, a' and e, in the same way as in the elliptic theory. It may be added that these third order terms are found in experience to be much smaller than those which arise in other ways.